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UNSOLVABILITY CORES IN CLASSIFICATION PROBLEMS

ULRIKE BRANDT AND HERMANN K.-G. WALTER

Fachbereich Informatik, Technische Universität Darmstadt, Germany
e-mail address: brandt@dekanat.informatik.tu-darmstadt.de
e-mail address: walter@informatik.tu-darmstadt.de

Abstract. Classification problems have been introduced by M. Ziegler as a generalization of promise problems. In this paper we are concerned with solvability and unsolvability questions with respect to a given set or language family, especially with cores of unsolvability. We generalize the results about unsolvability cores in promise problems to classification problems. Our main results are a characterization of unsolvability cores via cohesiveness and existence theorems for such cores in unsolvable classification problems. In contrast to promise problems we have to strengthen the conditions to assert the existence of such cores. In general unsolvable classification problems with more than two components exist, which possess no cores, even if the set family under consideration satisfies the assumptions which are necessary to prove the existence of cores in unsolvable promise problems. But, if one of the components is fixed we can use the results on unsolvability cores in promise problems, to assert the existence of such cores in general. In this case we speak of conditional classification problems and conditional cores. The existence of conditional cores can be related to complexity cores. Using this connection we can prove for language families, that conditional cores with recursive components exist, provided that this family admits an uniform solution for the word problem.

Introduction

The concept of *classsification problems* was introduced by M. Ziegler ([1]) as a generalization of *promise problems* due to S. Even ([5]). Promise problems are a generalization of *decision problems*. A classification problem is a vector $\mathbf{A} = (A_1, \ldots, A_k)$ where the A_i are pairwise disjoint infinite subsets of a given basic set S. For a set family $\mathcal{F} \subseteq \mathbf{2}^S$ such a classification problem is $\mathcal{F}$-*solvable*, if a vector $\mathbf{Q} = (Q_1, \ldots, Q_k)$ exists with $A_i \subseteq Q_i$, $Q_i \in \mathcal{F}, Q_i \cap Q_j = \emptyset$ for $1 \leq i \neq j \leq k$ and $Q_1 \cup \cdots \cup Q_k = S$. If $k = 2$ we are faced with promise problems. In applications $S = X^*$ where X is a finite nonempty alphabet and $\mathcal{F} = \mathcal{L}$ a language family and/or a complexity class. From an algorithmic point of view solutions of classification problems can be used to obtain constant size *advices*. In this case advices indicate the inputs to belong to certain subsets (c.f. [1] for further details). We extend the results about unsolvability cores in promise problems ([4]) to unsolvability cores in classification problems.

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Again cohesiveness is the characterizing indicator. For unsolvable promise problems we can find in general unsolvability cores, if the set family is closed under union, intersection and finite variation. But for unsolvable classification problems with $k > 2$ the existence of unsolvability cores needs further conditions. We show, that we can assert the existence of unsolvability cores for $k > 2$ under the same assumption as needed for promise problems, if we fix one of the components. In this approach the fixed component is called the *condition* for the classification problem. The results are proven under assumptions which involve closure properties of $\mathcal{F}$ against some or all boolean operations *union, intersection* and *complementation*. Moreover, we can relate unsolvability cores for conditional classification problems to so called *proper hard cores* introduced by R. Book and D.-Z. Du in a general form ([3]) and first defined by N. Lynch ([6]) for complexity classes. Using results and proof techniques from [3] we can apply our results to language families and complexity classes. Especially, we are able to construct unsolvability cores where the components are recursive. To do this, the language family or complexity class under consideration must allow an enumeration where the word problem has a uniform solution. We assume the reader to be familiar with the theory of recursive functions, languages and complexity (cf.[2],[7]).

1. Set and Language Families, Basic Notations

In the following an infinite *basic* set S is given. We assume that the elements of set families $\mathcal{F}$ are subsets of S. Moreover, sets $A, A', B, B', C, \cdots, Q, \cdots$ are always subsets of S and *singletons* $\{s\}$ are identified with s. We mainly deal with denumerable set families $\mathcal{F}$; i.e. a function $\mathbf{e}_{\mathcal{F}}: \mathbb{N}_0 \to \mathbf{2}^S$ with $\mathbf{e}_{\mathcal{F}}(\mathbb{N}_0) = \mathcal{F}$ exists (*enumeration* of $\mathcal{F}$). Consider the *boolean* operations $A \cup B$ *union*, $A \cap B$ *intersection* and $A^{\mathbf{c}} = S \backslash A$ *complementation* in connection with set families $\mathcal{F}$. These operations can be lifted to binary operations between set families $\mathcal{F}_1$ and $\mathcal{F}_2$ and unary operations for $\mathcal{F}$. Define

$$\mathcal{F}_1 \oplus \mathcal{F}_2 = \{A \cup B | A \in \mathcal{F}_1 \text{ and } B \in \mathcal{F}_2\},$$
$$\mathcal{F}_1 \odot \mathcal{F}_2 = \{A \cap B | A \in \mathcal{F}_1 \text{ and } B \in \mathcal{F}_2\}$$

and the closure operations

$$\mathcal{F}^{\mathbf{u}} = \{A_1 \cup \ldots \cup A_n | n \geq 1, A_i \in \mathcal{F} \ for \ 1 \leq i \leq n\}(union),$$
$$\mathcal{F}^{\mathbf{s}} = \{A_1 \cap \ldots \cap A_n | n \geq 1, A_i \in \mathcal{F} \ for \ 1 \leq i \leq n\}(intersection),$$
$$\mathcal{F}^{\mathbf{co}} = \{A^{\mathbf{c}} | A \in \mathcal{F}\}, \ \mathcal{F}^{\mathbf{cc}} = \mathcal{F} \cup \mathcal{F}^{\mathbf{co}}(complementation) \text{ and}$$
$$\mathcal{F}^{\mathbf{b}} = ((\mathcal{F}^{\mathbf{cc}})^{\mathbf{s}})^{\mathbf{u}}(boolean\ closure).$$

We will frequently use $\mathcal{F}^{\mathbf{dc}} = \mathcal{F} \cap \mathcal{F}^{\mathbf{co}}$. Note, that $(\mathcal{F}^{\mathbf{u}})^{\mathbf{s}} = (\mathcal{F}^{\mathbf{s}})^{\mathbf{u}}(distributivity), (\mathcal{F}^{\mathbf{co}})^{\mathbf{u}} = (\mathcal{F}^{\mathbf{s}})^{\mathbf{co}}(deMorgan)$, $(\mathcal{F}^{\mathbf{cc}})^{\mathbf{dc}} = \mathcal{F}^{\mathbf{cc}}$ and $(\mathcal{F}^{\mathbf{co}})^{\mathbf{co}} = \mathcal{F}$. Furthermore, $\mathcal{F} = \mathcal{F}^{\mathbf{cc}}$ ($\mathcal{F} = \mathcal{F}^{\mathbf{u}}$, $\mathcal{F} = \mathcal{F}^{\mathbf{s}}$) if and only if $\mathcal{F} = \mathcal{F}^{\mathbf{co}}$ ($\mathcal{F} \oplus \mathcal{F} \subseteq \mathcal{F}$, $\mathcal{F} \odot \mathcal{F} \subseteq \mathcal{F}$, respectively).

Let $\boldsymbol{fin}(S) = \{A \subseteq S | A \text{ finite}\}$. Then $\mathcal{F}$ is closed under *finite variation* if $\mathcal{F} \oplus \boldsymbol{fin}(S) \subseteq \mathcal{F}$ and $\mathcal{F} \odot \boldsymbol{fin}(S)^{\mathbf{co}} \subseteq \mathcal{F}$. We call $\mathcal{F}$ *nontrivial* if $\emptyset, S \in \mathcal{F}$ and $\mathcal{F}$ is closed under finite variation. In this case $\boldsymbol{fin}(S) \subseteq \mathcal{F}$. Note, that $\boldsymbol{fin}(S) = \boldsymbol{fin}(S)^{\mathbf{b}}$. Moreover, $\mathcal{F}^{\mathbf{cc}}$, $\mathcal{F}^{\mathbf{u}}$, $\mathcal{F}^{\mathbf{s}}$ and $\mathcal{F}^{\mathbf{b}}$ are nontrivial, if $\mathcal{F}$ is nontrivial.

Consider the case $S = X^*$, where X^* is the *free monoid over* X (a nonempty, finite *alphabet*) with *concatenation* of *words* as monoid operation and $\mathbf{1}$ as identity. As usual $L \subseteq X^*$ is called a *language* and $\mathcal{L} \subseteq \mathbf{2}^{X^*}$ a *language family*. For a word $w = x_1 \ldots x_n$ ($x_i \in X$ for $1 \leq i \leq n$) $|w| = n$ is the *length* of w and $|\mathbf{1}| = 0$. For languages L_1 and L_2 the *complex product* is defined by $L_1L_2 = \{w_1w_2|w_1 \in L_1, w_2 \in L_2\}$. There are various kinds of quotients available, for example the *left quotient* defined by $L_1^{-1}L_2 = \{w| \ \exists w_1 \in L_1\colon w_1w \in L_1\}$. In this context we are mainly interested in handling *leftmarkers*, i.e. we consider the products wL and the quotients $w^{-1}L$ where $w \in X^*$ and L is a language. With respect to language families $\mathcal{L}$ we get the closure operations $\mathcal{L}^{\mathbf{ltr}} = \{wL|w \in X^*, L \in \mathcal{L}\}$ and $\mathcal{L}^{\textbf{-ltr}} = \{w^{-1}L|w \in X^*, L \in \mathcal{L}\}$. In handling the leftmarkers (for example complementation of a leftmarked language) we use variation by $\mathcal{L}_{\mathbf{reg}}(X)$, the family of *regular* languages (for details see [4]). A language family $\mathcal{L}$ is closed under *regular variation* if $\mathcal{L} \oplus \mathcal{L}_{\mathbf{reg}}(X) \subseteq \mathcal{L}$ and $\mathcal{L} \odot \mathcal{L}_{\mathbf{reg}}(X) \subseteq \mathcal{L}$.

Looking at (partial) orderings on X^* the *lexicographic* ordering is important for our purposes. For $n \geq 0$ let $[n]_0 = \{0, \ldots, n-1\}$ and $[n] = \{1, \ldots, n\}$. Given a bijection $\omega : X \to [b]_0$ $(b = \#(X))$ define $w \leq v$ if and only if $(|w| < |v|$ or $(|w| = |v|$ and $(\forall u \in X^*, x, y \in X :\ w \in uxX^*$ and $v \in uyX^* \Rightarrow \omega(x) \leq \omega(y)))$. This is a well-ordering, hence we can define a successor function $\boldsymbol{succ}$ for $w \in X^*$ by $\boldsymbol{succ}(w) = \boldsymbol{min}\{v \in X^*|w \neq v$ and $w \leq v\}$ where the minimum is taken with respect to the lexicographic ordering. Then $\lambda i.\boldsymbol{lex}(i) = \boldsymbol{succ}^i(\mathbf{1})$ defines a bijection $\boldsymbol{lex} :\ \mathbb{N}_0 \to X^*$ with inverse $\boldsymbol{ord} = \boldsymbol{lex}^{-1}$.

Consider the language families $\mathcal{L}_{\mathbf{r.e.}}(X)$ (*recursively enumerable* languages) and $\mathcal{L}_{\mathbf{rec}}(X) = \mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{dc}}$ (*recursive* languages). Let $\boldsymbol{rec}_n (n \geq 0)$ be the set of *n-ary recursive* functions. Using $0, 1 \in \mathbb{N}_0$ as truth values define for a language L the function $\lambda i.\delta_L(i) =$ "$\boldsymbol{lex}(i) \in L$". Then a language L is recursive if and only if $\delta_L \in \boldsymbol{rec}_1$. Alternatively, a nonempty language L is recursive if and only if a function $f : \mathbb{N}_0 \to X^*$ exists such that $\lambda i.\boldsymbol{ord}(f(i))$ is nondecreasing and recursive. Classical language families and complexity classes are always denumerable. Of special interest are families with enumerations which are in a certain sense "effective". For our purpose it is important to assert that these enumerations allow a uniform solution for the word problem. More formular, we define for an enumeration $\mathbf{e}$ of a language family $\mathcal{L}$ the function $\lambda i, j.\boldsymbol{word}_{\mathbf{e}}(i, j) =$ "$\boldsymbol{lex}(j) \in \mathbf{e}(i)$". If $\boldsymbol{word}_{\mathbf{e}} \in \boldsymbol{rec}_2$ then $\mathbf{e}$ is called *WP-recursive*. $\mathcal{L}$ is called *WP-recursive*, if a WP-recursive enumeration $\mathbf{e}$ of $\mathcal{L}$ exists. Note, that any WP-recursive $\mathcal{L}$ is a (proper) subfamily of $\mathcal{L}_{\mathbf{rec}}(X)$ and every complexity class with reasonable ressource bounds (*time-* and *space-constructability* [2]) is WP-recursive.

2. Solvability of Classification Problems

Let $k > 0$. We consider vectors $\mathbf{A} = (A_1, \ldots, A_k)$ with $A_i \subseteq S$ for $1 \leq i \leq k$. To such an $\mathbf{A}$ we associate two functions $\boldsymbol{set}(\mathbf{A}) = A_1 \cup \cdots \cup A_k$ and $|\mathbf{A}| = k$. Moreover, if $\mathbf{B} = (B_1, \ldots, B_m)$ with $1 \leq m \leq k$ is another vector, then $\mathbf{B} \leq \mathbf{A}$ if and only if an injective $\sigma : [m] \to [k]$ exists with $B_i \subseteq A_{\sigma(i)}$ for $1 \leq i \leq m$. $\mathbf{A}$ is a *classification problem* if A_i is infinite and $A_i \cap A_j = \emptyset$ for all $1 \leq i \neq j \leq k$. For a given $\mathcal{F}$ a vector $\mathbf{Q} = (Q_1, \ldots, Q_k)$ is an $\mathcal{F}$ *-partition* if $\boldsymbol{set}(\mathbf{Q}) = S$, $Q_i \in \mathcal{F}$ and $Q_i \cap Q_j = \emptyset$ for $1 \leq i \neq j \leq k$.

Definition 2.1. A classification problem $\mathbf{A}$ is $\mathcal{F}$*-solvable* $(A \in \boldsymbol{class}_k(\mathcal{F}))$ if and only if an $\mathcal{F}$-partition $\mathbf{Q}$ exists with $|\mathbf{Q}| = k$ and $\mathbf{A} \leq \mathbf{Q}$, where $k = |\mathbf{A}|$.

If $S = \mathbb{N}_0$ then $\mathcal{F}$-solvability of promise problems corresponds to the *separation principle* defined in [7] (exercise 5-33). Our definition of $\mathcal{F}$-solvability for classification problems is stronger than the definition of $\mathcal{F}$*-separability* given in [1], where a classification problem $\mathbf{A}$ is $\mathcal{F}$*-separable*, if there exists a $\mathbf{Q}$, which satisfies the conditions of Definition 2.1 except the condition "$\boldsymbol{set}(\mathbf{Q}) = S$", which may not necessarily be valid. Note that for such a $\mathbf{Q}$, we always obtain $Q_k \subseteq (Q_1 \cup \cdots \cup Q_{k-1})^{\mathbf{c}}$. Hence, the class of $\mathcal{F}$-solvable classification problems with more than one components is identical with the class of $\mathcal{F}$-separable classification problems, if $\mathcal{F}$ is a boolean algebra. That $\mathcal{F}$-solvability is stronger than $\mathcal{F}$-separability, follows from results in [7]. Consider $\mathcal{L}_{\mathbf{r.e.}}(X)$ where X is a one-letter alphabet. Then a promise problem (A, B) consisting of recursively enumerable sets exists, which is not $\mathcal{L}_{\mathbf{r.e.}}(X)$-solvable ([7] exercise 5-34). But (A, B) is clearly $\mathcal{L}_{\mathbf{r.e.}}(X)$-separable. We also find the interesting result that any promise problem (A, B) with $A, B \in \mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{co}}$ is $\mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{co}}$-solvable ([7] exercise 5-33). Hence all promise problems, which are $\mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{co}}$-separable are $\mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{co}}$-solvable. But $\mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{co}}$ is not closed under complementation.

For $k = 1$ we identify A_1 with (A_1). If $\mathcal{F}$ is nontrivial then every A_1 is $\mathcal{F}$-solvable. If $k > 2$ and $\mathcal{F}$ satisfies appropriate closure properties, then we can reduce the question of solvability of classification problems to solvability of promise problems. Directly from the definition we get

Proposition 2.2. *If $\mathcal{F} = \mathcal{F}^{\boldsymbol{u}}$ then for all classification problems $\boldsymbol{A}$ and $\boldsymbol{B}$ with $\boldsymbol{B} \leq \boldsymbol{A}$ $\boldsymbol{A} \in \boldsymbol{class}_{|\boldsymbol{A}|}(\mathcal{F})$ implies $\boldsymbol{B} \in \boldsymbol{class}_{|\boldsymbol{B}|}(\mathcal{F})$.*

Proof. Suppose $\mathbf{B} \leq \mathbf{A} \leq \mathbf{Q}$ where $\mathbf{Q}$ is an $\mathcal{F}$-partition. Let $B = (B_1, \ldots, B_m), A = (A_1, \ldots, A_k)$ and $Q = (Q_1, \ldots, Q_k)$. Then we can assume without loss of generality $B_i \subseteq A_i \subseteq Q_i$ for all i. Consider P $= Q_1 \cup \cdots \cup Q_k$. Then $P^{\mathbf{c}} = Q_{m+1} \cup \cdots \cup Q_k \in \mathcal{F}$. Hence, $\mathbf{Q'} = (Q_1, \ldots, Q_{k-1}, Q_k \cup P^{\mathbf{c}})$ is an $\mathcal{F}$-partition with $\mathbf{B} \leq \mathbf{Q'}$. □

Lemma 2.3. *If $\mathcal{F} = \mathcal{F}^{\boldsymbol{u}} = \mathcal{F}^{\boldsymbol{s}}$ and $\boldsymbol{A} = (A_1, \ldots A_k)$ is a classification problem then $\boldsymbol{A} \in \boldsymbol{class}_k(\mathcal{F})$ if and only if $(A_i, A_j) \in \boldsymbol{class}_2(\mathcal{F})$ for all $1 \leq i \neq j \leq k$.*

Proof. The "if part" follows by Proposition 2.2. Suppose that $(A_i, A_j) \in \boldsymbol{class}_2(\mathcal{F})$ for $1 \leq i \neq j \leq k$. Now we proceed by induction over $|\mathbf{A}| = k$. If k = 2 nothing is to prove. Let $\mathbf{A} = (A_1, \ldots, A_{k+1})$ and suppose $(A_1, \ldots, A_k) \in \boldsymbol{class}_k(\mathcal{F})$. Then an $\mathcal{F}$-partition $\mathbf{Q'} = (Q'_1, \ldots, Q'_k)$ with $(A_1, \ldots, A_k) \leq \mathbf{Q'}$ exists. Assume without loss of generality $A_i \subseteq Q'_i$ for $1 \leq i \leq k$. On the other side $Q''_i \in \mathcal{F}^{\mathbf{dc}}$ exist with $A_i \subseteq Q''_i$ and $A_{k+1} \subseteq (Q''_i)^{\mathbf{c}}$ for $1 \leq i \leq k$. Consider $P = Q''_1 \cup \cdots \cup Q''_k$. Then $A_i \subseteq P \in \mathcal{F}$ for $1 \leq i \leq k$ and $P^{\mathbf{c}} = (Q''_1)^{\mathbf{c}} \cap \cdots \cap (Q''_k)^{\mathbf{c}} \in \mathcal{F}$ with $A_{k+1} \subseteq P^{\mathbf{c}}$. This shows $\mathbf{Q} = (Q'_1 \cap P, \ldots, Q'_k \cap P, P^{\mathbf{c}})$ is an $\mathcal{F}$-partition with $\mathbf{A} \leq \mathbf{Q}$. □

As indicated in the introduction we generalize the notion of a classification problem to conditional classification problems by fixing one component as condition. Consider $C \subseteq S$ and a classification problem $\mathbf{A}$. Then $(C, \mathbf{A})$ is a *conditional classification problem* if $C \cap \boldsymbol{set}(\mathbf{A}) = \emptyset$, referring to C as the *problem condition.* C could be finite, even empty. If $C^{\mathbf{c}}$ is finite, then no conditional classification problems $(C, \mathbf{A})$ exist.

Definition 2.4. A conditional classification problem $(C, \mathbf{A})$ is $\mathcal{F}$-*solvable* ($A \in \boldsymbol{cclass}_k(C, \mathcal{F})$) if and only if an $\mathcal{F}$-partition $\mathbf{Q} = (Q_0, Q_1, \ldots, Q_k)$ exists with $C \subseteq Q_0$ and $\mathbf{A} \leq (Q_1, \ldots, Q_k)$ where $k = |\mathbf{A}|$.

The following facts follow directly from the definition

Proposition 2.5. *Let $\mathcal{F}$ and $k > 0$ be given.*

(1) $C_1 \subseteq C_2 \subseteq S \Rightarrow \boldsymbol{cclass}_k(C_2, \mathcal{F}) \subseteq \boldsymbol{cclass}_k(C_1, \mathcal{F})$.
(2) $C^{\mathbf{c}} \in \boldsymbol{fin}(S) \Rightarrow \boldsymbol{cclass}_k(C, \mathcal{F}) = \emptyset$.
(3) $\emptyset \in \mathcal{F} \Rightarrow \boldsymbol{class}_k(\mathcal{F}) \subseteq \boldsymbol{cclass}_k(\emptyset, \mathcal{F})$.
(4) $\mathcal{F} = \mathcal{F}^{\boldsymbol{u}} \Rightarrow \boldsymbol{class}_k(\mathcal{F}) = \boldsymbol{cclass}_k(\emptyset, \mathcal{F})$.
(5) $\mathcal{F}$ nontrivial and $C \in \boldsymbol{fin}(S) \Rightarrow \boldsymbol{cclass}_k(C, \mathcal{F}) = \boldsymbol{cclass}_k(\emptyset, \mathcal{F})$.

Example 2.6. Consider $X = \{a, b\}$. Let $\mathcal{L} = \mathcal{L}^{\mathbf{ltr}} = \mathcal{L}^{\textbf{-ltr}}$ a nontrivial language family, which is closed under regular variation. If A is a set with $A^{\mathbf{c}}, A \notin \mathcal{L}$, then $(A^{\mathbf{c}}, A) \notin \boldsymbol{class}_2(\mathcal{L})$ and by our assumption on $\mathcal{L}$ $(xA^{\mathbf{c}}, xA) \notin \boldsymbol{class}_2(\mathcal{L})$ for $x = a, b$ (Lemma 5.4 in [4]). Clearly, $(aA^{\mathbf{c}}, bA) \in \boldsymbol{class}_2(\mathcal{L})$, but $(aA \cup bA^{\mathbf{c}}, aA^{\mathbf{c}}, bA) \notin \boldsymbol{class}_3(\mathcal{L})$. Hence $(aA^{\mathbf{c}}, bA) \notin \boldsymbol{cclass}_2(aA \cup bA^{\mathbf{c}}, \mathcal{L})$.

3. Unsolvability Cores in Classification Problems

As in the case of promise problems unsolvability of classification problems is closely related to cohesiveness.

Definition 3.1. $A \subseteq S$ is $\mathcal{F}$-cohesive ($A \in \boldsymbol{cohesive}(\mathcal{F})$) if and only if A is infinite and for all $Q \in \mathcal{F}^{\mathbf{dc}}$ either $A \cap Q$ or $A \cap Q^{\mathbf{c}}$ is finite (cf.[4] and [7]).

In [4] (Theorem 5.1.) it is proven, that for a promise problem (A, B) and a nontrivial set family $\mathcal{F}$ $A \cup B \in \mathbf{cohesive}(\mathcal{F})$ if and only if $A, B \in \mathbf{cohesive}(\mathcal{F})$ and $(A, B) \notin \mathbf{class}_2(\mathcal{F})$. This result leads to a much stronger one. In the theory of complexity we find the notion of *hard cores* inside those sets which can be computed with bounded ressources (time, space, e.t.c. [3]). Similarily, we can consider *unsolvability cores* of classification problems which are not solvable.

Definition 3.2. For $k > 1$ a classification problem $\mathbf{A}$ with $|\mathbf{A}| = k$ is a *k-core* of $\mathcal{F}$ ($\mathbf{A} \in \boldsymbol{core}_k(\mathcal{F})$) if and only if for all classification problems $\mathbf{A}'$ with $\mathbf{A}' \leq \mathbf{A}$ and $|\mathbf{A}'| > 1$: $\mathbf{A}' \notin \boldsymbol{class}_{|\mathbf{A}'|}(\mathcal{F})$.

Clearly, any subproblem of a core is itself a core. This is especially true for subproblems, which are promise problems. This enables us to use the results about unsolvability cores for promise problems from [4].

Lemma 3.3. *If $\mathcal{F} = \mathcal{F}^{\boldsymbol{u}}$ and $\boldsymbol{A} = (A_1, \ldots, A_k)(k > 1)$ is a classification problem then $\boldsymbol{A} \in \boldsymbol{core}_k(\mathcal{F})$ if and only if $(A_i, A_j) \in \boldsymbol{core}_2(\mathcal{F})$ for all $1 \leq i \neq j \leq k$.*

Proof. Suppose $\mathbf{A} \in \boldsymbol{core}_k(\mathcal{F})$, then by definition $(A_i, A_j) \leq \mathbf{A}$ and therefore $(A_i, A_j) \in \boldsymbol{core}_2(\mathcal{F})$. Conversely, suppose that $\mathbf{A} \notin \boldsymbol{core}_k(\mathcal{F})$, i. e. $\mathbf{A}' = (A'_1, \ldots, A'_m)$ exists with $\mathbf{A}' \leq \mathbf{A}$, $m > 1$ and $\mathbf{A}' \in \boldsymbol{class}_{|\mathbf{A}'|}(\mathcal{F})$. Since $\mathcal{F} = \mathcal{F}^{\mathbf{u}}$ we know $(A'_1, A'_2) \in \boldsymbol{class}_2(\mathcal{F})$. Moreover, $A'_1 \subseteq A_i$ and $A'_2 \subseteq A_j$ for some $1 \leq i \neq j \leq k$. But then $(A_i, A_j) \notin \boldsymbol{core}_2(\mathcal{F})$. □

Now we can characterize cores by cohesiveness. Using Theorem 5.1. and Theorem 6.7. of [4] we can prove

Theorem 3.4. *If $\mathcal{F} = \mathcal{F}^{\boldsymbol{u}}$ is nontrivial and $\boldsymbol{A}$ a classification problem with $|\boldsymbol{A}| = k > 1$ then $\boldsymbol{A} \in \boldsymbol{core}_k(\mathcal{F})$ if and only if $\boldsymbol{set}(\boldsymbol{A}) \in \boldsymbol{cohesive}(\mathcal{F})$.*

Proof. If $\mathbf{A} = (A_1, \ldots, A_k) \in \boldsymbol{core}_k(\mathcal{F})$, then $(A_i, A_j) \in \boldsymbol{core}_2(\mathcal{F})$ for all $1 \leq i \neq j \leq k$. By Theorem 6.7. in [4] we know $A_1 \cup A_i \in \boldsymbol{cohesive}(\mathcal{F})$ for all $2 \leq i \leq k$. But then $A_1 \cup \cdots \cup A_k = (A_1 \cup A_2) \cup \cdots \cup (A_1 \cup A_k)$. Since $A_1 \subseteq (A_1 \cup A_i) \cap (A_1 \cup A_j)$ for all $2 \leq i \neq j \leq k$ and A_1 is infinite, a simple induction proof shows $\boldsymbol{set}(\mathbf{A}) \in \boldsymbol{cohesive}(\mathcal{F})$.

Conversely, if $A_1 \cup \cdots \cup A_k \in \boldsymbol{cohesive}(\mathcal{F})$ then for all $1 \leq i \neq j \leq k$, $A_i \cup A_j \in \boldsymbol{cohesive}(\mathcal{F})$. Again by Theorem 6.7. of [4] $(A_i, A_j) \in \boldsymbol{core}_2(\mathcal{F})$ and therefore by Lemma 3.3. $\mathbf{A} \in \boldsymbol{core}_k(\mathcal{F})$. □

We can find to any classification problem $\mathbf{A}$ with $|\mathbf{A}| = 2$ and $\mathbf{A} \notin \boldsymbol{class}_2(\mathcal{F})$ a $\mathbf{B} \leq \mathbf{A}$ such that $\mathbf{B} \in \boldsymbol{core}_2(\mathcal{F})$ if $\mathcal{F} = \mathcal{F}^{\mathbf{u}} = \mathcal{F}^{\mathbf{s}}$ is denumerable ([4]). But this is not true for classification problems $\mathbf{A}$ with $|\mathbf{A}| > 2$. To see this we prove the following theorem, where we use $S = X^*$ with $X = \{a, b, c\}$. Define for $A \subseteq X^*$ the classification problem $\mathbf{C}(A) = (A_{ab}, A_{bc}, A_{ca})$, where $A_{xy} = xA \cup yA^{\mathbf{c}}$ for $x, y \in X$.

Theorem 3.5. *Let $\mathcal{L}$ be a nontrivial language family with $\mathcal{L} = \mathcal{L}^{\mathbf{u}} = \mathcal{L}^{\boldsymbol{ltr}} = \mathcal{L}^{\textbf{-}\boldsymbol{ltr}}$, which is closed under regular variation. If $A \subseteq S$ with $A \notin \mathcal{L}$ or $A^{\mathbf{c}} \notin \mathcal{L}$, then $\boldsymbol{C}(A) \notin \boldsymbol{class}_3(\mathcal{L})$ and for all $\boldsymbol{B} \leq \boldsymbol{C}(A)$ with $|\boldsymbol{B}| = 3$: $\boldsymbol{B} \notin \boldsymbol{core}_3(\mathcal{L})$.*

Proof. (1) We know $(A^{\mathbf{c}}, A) \notin \boldsymbol{class}_2(\mathcal{L})$ ([4]). But then by Lemma 5.4 of [4] $(xA^{\mathbf{c}}, xA) \notin \boldsymbol{class}_2(\mathcal{L})$ for all $x \in X$. Now $(bA^{\mathbf{c}}, bA) \leq (A_{ab}, A_{bc})$, $(cA^{\mathbf{c}}, cA) \leq (A_{bc}, A_{ca})$ and $(aA^{\mathbf{c}}, aA) \leq (A_{ca}, A_{ab})$. This shows $(A_{xy}, A_{xz}) \notin \boldsymbol{class}_2(\mathcal{L})$ for all $x \neq y$, $z \neq y$ and $x \neq z$.

(2) Suppose $\mathbf{B} \leq \mathbf{C}(A)$ exists with $\mathbf{B} \in \boldsymbol{core}_3(\mathcal{L})$. Then by Theorem 3.4 $\boldsymbol{set}(\mathbf{B}) \in \boldsymbol{cohesive}(\mathcal{L})$. Assume without loss of generality that $\mathbf{B} = (B(a, b), B(b, c), B(c, a))$ and $B(x, y) \subseteq A_{xy}$ for $x, y \in X$ with $x \neq y$.. In the following let $B'(x, y) = B(x, y) \cap xX^*$ and $B''(x, y) = B(x, y) \cap (xX^*)^{\mathbf{c}}$.

Assertion : $B'(x, y) \in \boldsymbol{fin}(X^*)$ for all $x, y \in X$ with $x \neq y$.
Suppose to the contrary (without loss of generality) $B'(a, b) \notin \boldsymbol{fin}(X^*)$. But then $B'(b, c) \in \boldsymbol{fin}(X^*)$. Otherwise we obtain $(B'(a, b), B'(b, c)) \leq (aX^*, bX^*) \leq (aX^*, (aX^*)^{\mathbf{c}})$. Since $\mathcal{L}_{\mathbf{reg}} \subseteq \mathcal{L}$, $\mathbf{B} \notin \boldsymbol{core}_3(\mathcal{L})$ - a contradiction. But now $B''(b, c)$ is infinite and $B''(b, c) \subseteq cX^* \subseteq (aX^*)^{\mathbf{c}}$, hence both $\boldsymbol{set}(\mathbf{B}) \cap aX^*$ and $\boldsymbol{set}(\mathbf{B}) \cap (aX^*)^{\mathbf{c}}$ are infinite - a contradiction to $\boldsymbol{set}(\mathbf{B}) \in \boldsymbol{cohesive}(\mathcal{L})$.
Now consider $B''(a, b)$ and $B''(c, a)$. Then both sets are infinite and $(B''(a, b), B''(c, a)) \leq (bX^*, aX^*) \leq (bX^*, (bX^*)^{\mathbf{c}})$ - a contradiction to $\mathbf{B} \in \boldsymbol{core}_3(\mathcal{L})$. This completes the proof. □

Remark 3.6. The basic idea behind the proof of Theorem 3.5. is due to M. Ziegler ([1]). Note, that complexity classes and most of the known language families satisfy the conditions of Theorem 3.5.

Using conditional unsolvability, we can derive an existence theorem for cores.

Theorem 3.7. *Let $\mathcal{F} = \mathcal{F}^{\boldsymbol{u}} = \mathcal{F}^{\boldsymbol{s}}$ be denumerable and nontrivial. If $\boldsymbol{A} = (A_1, \ldots, A_k)$ is a classification problem and $C \subseteq \boldsymbol{set}(\boldsymbol{A})^{\boldsymbol{c}}$ is $\mathcal{F}$-cohesive with $(C, A_i) \notin \boldsymbol{class}_2(\mathcal{F})$ for $1 \leq i \leq k$, then there exists $\boldsymbol{B} \leq \boldsymbol{A}$ with $|\boldsymbol{B}| = k$ and $\boldsymbol{B} \in \boldsymbol{core}_k(\mathcal{F})$.*

Proof. Since $(C, A_i) \notin \boldsymbol{class}_2(\mathcal{F})$, we can find $C_i \subseteq C$ and $B_i \subseteq A_i$ with $(C_i, B_i) \in \boldsymbol{core}_2(\mathcal{F})$ (Theorem 6.14 in [4]). By Theorem 3.5. $C_i \cup B_i \in \boldsymbol{cohesive}(\mathcal{F})$ and therefore $B_i \in \boldsymbol{cohesive}(\mathcal{F})$. Now $(C, B_i) \notin \boldsymbol{class}_2(\mathcal{F})$ and $C \in \boldsymbol{cohesive}(\mathcal{F})$. By Theorem 5.1. in [4] we know $C \cup B_i \in \boldsymbol{cohesive}(\mathcal{F})$. But then $C \cup B_1 \cup \cdots \cup B_k = (C \cup B_1) \cup \cdots \cup (C \cup B_k) \in \boldsymbol{cohesive}(\mathcal{F})$, since for all $1 \leq i \neq j \leq k$ C is infinite and $C \subseteq (C \cup B_i) \cap (C \cup B_j)$. It follows $B_1 \cup \cdots \cup B_k \in \boldsymbol{cohesive}(\mathcal{F})$ and we obtain $\mathbf{B} = (B_1, \ldots, B_k) \leq \mathbf{A}$ and by Theorem 3.4 $\mathbf{B} \in \boldsymbol{core}_k(\mathcal{F})$. □

Remark 3.8. Consider the situation of Theorem 3.5. Then $\boldsymbol{set}(\mathbf{C}(A)) = XX^*$ and there is no room for an infinite condition C to make the conditional classification problem $(C, \mathbf{C}(A))$ $\mathcal{L}$-solvable.

4. Cores in Conditional Classification Problems

Unsolvability of conditional classification problems can be related to cohesiveness, too.

Definition 4.1. Let $C, A \subseteq S$. Then A is *$\mathcal{F}$-cohesive under condition C* (in short: $A \in \boldsymbol{ccohesive}(C, \mathcal{F})$), if and only if A is infinite and for all $Q \in \mathcal{F}^{\mathbf{dc}}$ with $Q \subseteq C$ either $A \cap Q$ or $A \cap Q^c$ is finite.

Clearly, if $C_1 \subseteq C_2 \subseteq S$, then $\boldsymbol{ccohesive}(C_2, \mathcal{F}) \subseteq \boldsymbol{ccohesive}(C_1, \mathcal{F})$. Especially, we get $\boldsymbol{ccohesive}(S, \mathcal{F}) = \boldsymbol{cohesive}(\mathcal{F})$ and therefore $\boldsymbol{cohesive}(\mathcal{F}) \subseteq \boldsymbol{ccohesive}(C, \mathcal{F})$ for all $C \subseteq S$. Rewriting the definition, we also find $\boldsymbol{ccohesive}(C, \mathcal{F})) = \boldsymbol{cohesive}(\mathcal{F}(C)^{\mathbf{cc}})$ where $\mathcal{F}(C) = \{Q |\ Q \subseteq C \text{ and } Q \in \mathcal{F}\}$. Analogously, we define conditional cores by

Definition 4.2. Let $C \subseteq S$ and $\mathbf{A}$ a classification problem. Then $\mathbf{A}$ is a *C-conditional core* of $\mathcal{F}$ ($\mathbf{A} \in \boldsymbol{ccore}_{|\mathbf{A}|}(C, \mathcal{F})$) if and only if for all $\mathbf{A}' \leq \mathbf{A}$ with $|\mathbf{A}'| > 0 : \mathbf{A}' \notin \boldsymbol{cclass}_{|\mathbf{A}'|}(C, \mathcal{F})$.

In contrast to the definition of $\boldsymbol{core}(\mathcal{F})$ subproblems $\mathbf{A}'$ with $|\mathbf{A}'| = 1$ are considered, too. Note, that $(C, \mathbf{A}')$ is a conditional-classification problem, if $\mathbf{A}' \leq \mathbf{A}$. Moreover, if $\mathbf{A} \in \boldsymbol{ccore}_{|\mathbf{A}|}(C, \mathcal{F})$, then $\mathbf{A}' \in \boldsymbol{ccore}_{|\mathbf{A}'|}(C, \mathcal{F})$. The following lemma characterize $A \in \boldsymbol{ccore}_1(C, \mathcal{F})$ by conditional cohesiveness.

Lemma 4.3. *Let $\mathcal{F}$ be nontrivial and $C, A \subseteq S$ with A infinite and $A \cap C = \emptyset$. Then the following statements are equivalent*

(i) $A \in \boldsymbol{ccore}_1(C, \mathcal{F})$

(ii) $A \notin \boldsymbol{cclass}_1(C, \mathcal{F})$ *and* $A \in \boldsymbol{ccohesive}(C^c, \mathcal{F})$.

Proof. (i) $\Rightarrow$ (ii): Suppose $A \in \boldsymbol{ccore}_1(C, \mathcal{F})$. Then $A \notin \boldsymbol{cclass}_1(C, \mathcal{F})$. Assume to the contrary that $A \notin \boldsymbol{ccohesive}(C^{\mathbf{c}}, \mathcal{F})$. Then $Q \in \mathcal{F}^{\mathbf{dc}}$ exists with $Q \subseteq C^{\mathbf{c}}, A \cap Q \notin \boldsymbol{fin}(S)$ and $A \cap Q^{\mathbf{c}} \notin \boldsymbol{fin}(S)$. Let $B = A \cap Q$. Then $B \subseteq Q$, but $Q \subseteq C^{\mathbf{c}}$, hence $C \subseteq Q^{\mathbf{c}}$. Moreover, $Q, Q^{\mathbf{c}} \in \mathcal{F}$, i.e. $B \in \boldsymbol{cclass}_1(C, \mathcal{F})$.

(ii) $\Rightarrow$ (i): Suppose that $A \notin \boldsymbol{cclass}_1(C, \mathcal{F})$ and $A \in \boldsymbol{ccohesive}(C^{\mathbf{c}}, \mathcal{F})$. Assume to the contrary that an infinite set $B \subseteq A$ exists, such that $B \subseteq Q^{\mathbf{c}}$ and $C \subseteq Q$ for some $Q \in \mathcal{F}^{\mathbf{dc}}$. Then $Q^{\mathbf{c}} \subseteq C^{\mathbf{c}}$. Since $B \cap Q^{\mathbf{c}} \notin \boldsymbol{fin}(S)$, $A \cap Q^{\mathbf{c}} \notin \boldsymbol{fin}(S)$, too. Hence $A \cap Q \in \boldsymbol{fin}(S)$, because $A \in \boldsymbol{ccohesive}(C^{\mathbf{c}}, \mathcal{F})$. Consider $Q' = Q^{\mathbf{c}} \cup (A \cap Q)$. Since $\mathcal{F}$ is nontrivial, $Q' \in \mathcal{F}$. Note that $A = (A \cap Q) \cup (A \cap Q^{\mathbf{c}}) \subseteq Q^{\mathbf{c}} \cup (A \cap Q) = Q'$. On the other side, $Q^{\mathbf{c}} \subseteq C^c$ and $A \cap Q \subseteq A \subseteq C^{\mathbf{c}}$, i.e. $Q' \subseteq C^{\mathbf{c}}$. Hence $C \subseteq Q'^{\mathbf{c}}$. This shows that $A \notin \boldsymbol{cclass}_1(C, \mathcal{F})$ - a contradiction. □

Theorem 4.4. *Let $\mathcal{F}$ be nontrivial with $\mathcal{F} = \mathcal{F}^{\boldsymbol{u}}$ and $(C, \boldsymbol{A})$ a conditional k-classification problem. If $\boldsymbol{A} = (A_1, \ldots, A_k)$ then the following statements are equivalent*

(i) $\boldsymbol{A} \in \boldsymbol{ccore}_k(C, \mathcal{F})$

(ii) $A_i \notin \boldsymbol{cclass}_1(C, \mathcal{F})$ *and* $A_i \in \boldsymbol{ccohesive}(C^{\boldsymbol{c}}, \mathcal{F})$ *for all* $1 \leq i \leq k$.

Proof. (i) $\Rightarrow$ (ii): Suppose that $\mathbf{A} \in \boldsymbol{ccore}_k(C, \mathcal{F})$. Then for all $1 \leq i \leq k : (C, A_i) \in \boldsymbol{ccore}_1(C, \mathcal{F})$, since $A_i \leq \mathbf{A}$. Applying Lemma 4.3. we get the result.

(ii) $\Rightarrow$ (i): Let the A_i be given according to the assumption. Assume to the contrary that $\mathbf{B} \leq \mathbf{A}$ exists with $\mathbf{B} = (B_1, \ldots, B_m) \in \boldsymbol{cclass}_1(C, \mathcal{F})$. Then an injective $\sigma : [m] \to [k]$ exists with $B_i \subseteq A_{\sigma(i)}$ for $1 \leq i \leq k$. Since $\mathcal{F} = \mathcal{F}^{\mathbf{u}}, B_i \in \boldsymbol{cclass}_1(C, \mathcal{F})$. But $A_{\sigma(i)} \in \boldsymbol{core}_1(C, \mathcal{F})$ and $B_i \subseteq A_{\sigma(i)}$. This is a contradiction. □

Now, we are able to assert the existence of conditional cores in the case that both C and $C^{\mathbf{c}}$ are infinite. Observe that under this assumption $A \in \boldsymbol{cclass}_1(C, \mathcal{F})$ if and only if (C, A) considered as a promise problem is solvable for $\mathcal{F}$, i.e. $(C, A) \in \boldsymbol{class}_1(\mathcal{F})$.

Lemma 4.5. *Let $\mathcal{F}$ be denumerable and nontrivial with $\mathcal{F} = \mathcal{F}^{\boldsymbol{u}} = \mathcal{F}^{\boldsymbol{s}}$. If $A \notin \boldsymbol{fin}(S)$, $C \notin \boldsymbol{fin}(S)^{\boldsymbol{cc}}$, $A \cap C = \emptyset$ and $A \notin \boldsymbol{cclass}_1(C, \mathcal{F})$, then $B \subseteq A$ exists with $B \in \boldsymbol{ccore}_1(C, \mathcal{F})$.*

Proof. If $A \notin \boldsymbol{cclass}_1(C, \mathcal{F})$, i.e. $(C, A) \notin \boldsymbol{class}_1(\mathcal{F})$. By cor.6.17. in[4] we can find $B \subseteq A$ such that for all infinite $B' \subseteq B$ $(C, B') \notin \boldsymbol{class}_2(\mathcal{F})$, i.e. $B \in \boldsymbol{ccore}_1(C, \mathcal{F})$. □

Using this lemma in connection with Theorem 4.4. we get

Lemma 4.6. *Let $\mathcal{F}$ be denumerable and nontrivial with $\mathcal{F} = \mathcal{F}^{\boldsymbol{u}} = \mathcal{F}^{\boldsymbol{s}}$ and $(C, \boldsymbol{A})$ a conditional classification problem where C and $C^{\mathbf{c}}$ are infinite. If $\boldsymbol{A} = (A_1, \ldots, A_k)$ with $A_i \notin \boldsymbol{cclass}_1(C, \mathcal{F})$ for $1 \leq i \leq k$ then a $\boldsymbol{B} \leq \boldsymbol{A}$ exists with $|\boldsymbol{B}| = k$ and $\boldsymbol{B} \in \boldsymbol{ccore}_k(C, \mathcal{F})$.*

Proof. By Lemma 4.5. we find for each $1 \leq i \leq k$ $B_i \in \boldsymbol{ccore}_1(C, \mathcal{F})$ and $B_i \subseteq A_i$. Let $\mathbf{B} = (B_1, \ldots, B_k)$. Then $\mathbf{B} \leq \mathbf{A}$ and $|\mathbf{B}| = k$. By Theorem 4.4. $\mathbf{B} \in \boldsymbol{ccore}_k(C, \mathcal{F})$. □

5. Conditional Cores and Hard Cores

For WP-recursive language families we can prove a much stronger result. This depends on the relation between $A \in \boldsymbol{ccore}_1(C, \mathcal{F})$ and proper hard cores introduced by N. Lynch [6] for complexity classes and in a very general form by R. Book- D.-Z. Du [3].

Definition 5.1. B is a *$\mathcal{F}$-hardcore* of A if and only if B is infinite and for all $C \in \mathcal{F}(A)$: $B \cap C \in \boldsymbol{fin}(S)$. If additionally $B \subseteq A$ then B is a *proper* $\mathcal{F}$-hardcore of A. (Remind $\mathcal{F}(A) = \{Q \subseteq A \mid Q \in \mathcal{F}\}$ for $\mathcal{F}$ and A.)

Note, that for $A' \subseteq A$ with A' infinite every $\mathcal{F}$-hardcore of A is a $\mathcal{F}$-hardcore of A'. Rephrasing Lemma 7.2 of [4] we get the following

Lemma 5.2. *If $\mathcal{F}$ is nontrivial with $\mathcal{F} = \mathcal{F}^{\boldsymbol{co}}$ and (C, A) a conditional classification problem then A is a proper $\mathcal{F}$-hardcore of $C^{\mathbf{c}}$ if and only if $A \in \boldsymbol{ccore}_1(C, \mathcal{F})$.*

Now we can use a construction for proper hard cores from [3] in a modified form.

Theorem 5.3. *If $\mathcal{L}$ is a nontrivial and WP-recursive language family with $\mathcal{L} = \mathcal{L}^{\boldsymbol{b}}$ and (C, A) a conditional classification problem with $A \notin \boldsymbol{cclass}_1(C, \mathcal{L})$ and C, A are recursive then a recursive $B \subseteq A$ exists with $B \in \boldsymbol{ccore}_1(C, \mathcal{L})$.*

Proof. Consider an enumeration $\mathbf{e}$ of $\mathcal{L}$ such that $\boldsymbol{word}_{\mathbf{e}} \in \boldsymbol{rec}_2$. Furthermore, let $\delta_C, \delta_A \in \boldsymbol{rec}_1$. Now define for all $n \geq 0$ $B(n)$, cancel(n) and card(n) by the following algorithm:

if $\boldsymbol{lex}(0) \in C$ **then**
  cancel$(0) := 0$
end if
if $\boldsymbol{lex}(0) \in A$ and $\boldsymbol{lex} \notin \mathbf{e}(0)$ **then**
  $B(0) := 0$; card$(0) := 1$
end if
$n := 1$;
while $n \neq 0$ **do**

$B(n) := B(n-1)$; $\text{cancel}(n) := \text{cancel}(n-1)$; $\text{card}(n) := \text{card}(n-1)$;
if $\boldsymbol{lex}(n) \in C$ **then**
  $\text{cancel}(n) := \text{cancel}(n) \cup \{i | 0 \leq i \leq \text{card}(n) \text{ and } \boldsymbol{lex}(n) \in \mathbf{e}(i)\}$
end if
if $\boldsymbol{lex}(n) \in A$ and $\forall\, 0 \leq i \leq \text{card}(n) : (i \notin \text{cancel}(n) \Rightarrow \boldsymbol{lex}(n) \notin \mathbf{e}(i))$ **then**
  $B(n) := B(n) \cup \boldsymbol{lex}(n); \text{card}(n) := \text{card}(n) + 1$
end if;
$n := n + 1$
end while

(For $A = C^{\mathbf{c}}$ we get the construction of [3]).

Now, let $B = \bigcup_{i=0}^{\infty} B(n)$ and $\text{cancel} = \bigcup_{i=0}^{\infty} \text{cancel}(i)$. Assume for the moment that B is infinite. B is recursive and $B \subseteq A$, since all basic functions are recursive, $\text{cancel}(n)$ is finite for all n and the elements of B are added in increasing order with respect to $\boldsymbol{lex}$. Moreover, $\lim_{n\to\infty}\text{card}(n) = \infty$. Hence $\{k | \mathbf{e}(k) \cap C \neq \emptyset\} = \text{cancel}$ and we get $\mathbf{e}(i) \subseteq C^{\mathbf{c}}$ and by construction $\mathbf{e}(i) \cap B \in \boldsymbol{fin}(X^*)$ for $i \notin \text{cancel}$ (cf. [3]). In conclusion, B is a proper $\mathcal{L}$-hardcore of $C^{\mathbf{c}}$ and by Lemma 4.9. $B \in \boldsymbol{ccore}_1(C, \mathcal{L})$. It remains to show the

Assertion: $B \notin \boldsymbol{fin}(X^*)$.

Suppose to the contrary, that B is finite. Then M exists with $card(n) = M$ for almost all n. Moreover, for every $i \in [M+1]_0$ with $\mathbf{e}(i) \cap C \neq \emptyset$ there must exist $K(i)$ with $i \in \text{cancel}(K(i))$. Let $K = \boldsymbol{max}\{K(i) | i \in [M+1]_0 \text{ with } \mathbf{e}(i) \cap C \neq \emptyset\}$. Then we know that for all $i \in [M+1]_0$ with $i \notin \text{cancel}(K(i)) : \mathbf{e}(i) \subseteq C^{\mathbf{c}}$. Choose $N \geq K$ sufficiently large such that additionally $\text{card}(n) = M$ for every $n \geq N$. Consider $\boldsymbol{lex}(n) \in A$ with $n \geq N$. Since $\boldsymbol{lex}(n) \notin B$, $i \in [M+1]_0$ exists with $\boldsymbol{lex}(n) \in \mathbf{e}(i)$. This shows $A \subseteq \{\boldsymbol{lex}(k) | k < N$ and $\boldsymbol{lex}(k) \in A\} \cup \bigcup_{i=0, i\notin\text{cancel}}^{M} \mathbf{e}(i) = Q \subseteq C^{\mathbf{c}}$ and therefore $C \subseteq Q^{\mathbf{c}}$. Since $\mathcal{L}$ is nontrivial and $\mathcal{L} = \mathcal{L}^{\mathbf{u}}$, we know $Q \in \mathcal{L}$. Moreover, $\mathcal{L} = \mathcal{L}^{\mathbf{co}}$ implies $Q^{\mathbf{c}} \in \mathcal{L}$, hence $A \notin \boldsymbol{cclass}_1(C, \mathcal{L})$ - a contradiction. □

Now we can derive a stronger result than Lemma 4.6.:

Theorem 5.4. *Let $\mathcal{L}$ be a nontrivial and WP-recursive language family with $\mathcal{L} = \mathcal{L}^{\boldsymbol{b}}$ and $(C, \boldsymbol{A})$ a conditional k-classification problem. If C is recursive and $\boldsymbol{A} = (A_1, \ldots, A_k)$ such that $A_i \in \boldsymbol{cclass}_1(C, \mathcal{L})$ and A_i is recursive for $1 \leq i \leq k$ then $\boldsymbol{B} = (B_1, \ldots, B_k)$ exists with $\boldsymbol{B} \leq \boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{ccore}_k(C, \mathcal{L})$ and B_i is recursive for $1 \leq i \leq k$.*

Proof. By Theorem 5.3. we find for each $1 \leq i \leq k$ $B_i \in \boldsymbol{cclass}_1(C, \mathcal{L})$ with $B_i \subseteq A_i$ and B_i is recursive. Let $\mathbf{B} = (B_1, \ldots, B_k)$. Then $\mathbf{B} \leq \mathbf{A}$ and by Theorem 4.4. $\mathbf{B} \in \boldsymbol{ccore}_k(C, \mathcal{L})$. □

Remark 5.5. The B_i's constructed in Theorem 5.4. are all infinite. By the Dekker-Myhill theorem (12.3 Theorem VI in [7]), we can find in every B_i a $\mathcal{L}$-cohesive B_i', but we cannot show, that B_i' is recursive under the conditions of Theorem 5.4. The best result to our knowledge is the result of Friedberg (§12.4 Theorem XI in [7]). The construction (due to Yates) in the proof given in [7] can be easily modified in such a way, that to any infinite, recursive A a $\mathcal{L}_{\mathbf{r.e.}}(X)$-cohesive subset B with $B^{\mathbf{c}} \in \mathcal{L}_{\mathbf{r.e.}}(X)$ can be found. Since any WP-recursive language family $\mathcal{L}$ is a subfamily of $\mathcal{L}_{\mathbf{r.e.}}(X)$ this B is $\mathcal{L}$-cohesive, too.

Concluding Remarks

This paper continues our research about unsolvability cores in promise problems ([4]) generalizing the results to classification problems. Our approach is very general, though the applications in this paper deal mainly with language families and complexity classes. The main open problem in our approach is to construct cohesive sets with "nice" properties.

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Referee no 2:

* The paper can be accepted for Logical Methods in Computer Science after minor revisions

Promise problems are a generalization of languages as decision problems in that the behavior (terminate, output yes/no) of an algorithmic solution is prescribed only on certain subset of all possible input strings. Classification problems extend this further from Boolean (yes/no) to a fixed number k>2 of possible answers.

A (hard) core of a classical decision problem L is essentially a sub-problem all whose infinite sub-problems are algorithmically difficult to solve. Previous work [4] has generalized this concept to promise problems and explored its properties and in particular existence; and the present submission extends these considerations to classification problems.

The approach taken by the authors is impressively general: Fixing a class F of languages to define 'easy' leads to generic Definitions 2.1 (F-solvable classification problem), 3.1 (F-cohesive), 3.3 (F-core) and so on. This covers all practical cases such as F=P (polynomial-time decidable languages), PSPACE, EXP etc. as well as for instance levels 3 and 1 of Chomsky's hierarchy.

Of course some conditions have to be imposed on F, such as closure under binary unions/intersections, complements, finite variation, prefixes/suffixes, containing all regular languages, or being effectively enumerable (WP-recursive, end of Section 1): The authors take particular care in their results to invoke as few of these prerequisites as possible.

However Theorem 3.6 demonstrates that even under strong hypothesis classification (as opposed to promise) problems do not contain cores; which leads the paper to consider _conditional_ classification problems (Section 4, in particular Lemma 4.6) and hard cores in the sense of Book&Du (Section 5, in particular Theorem 5.3).

The paper fits well into the scope of LMCS. It is carefully and concisely written, and I recommend publication subject to only few minor comments:

Intro l.4: "basic (usually infinite) set S": remove (usually infinite) done

S.1 l.1: "and we assume for set families F" ?? text revised

p.2 l.-7: L_{1,2} -> L_1,L_2 done

p.2 l.-1: "Handling the leftmarkers we make use of another kind of variation": Could not make sense of that, sorry. text revised

p.3 l.4: [n] = [1...n] or {1,...,n} ? it is the same, we corrected to the second form

p.3 l.6: Everybody "knows" what the lexicographic ordering is, but your formalization differs from that and in fact fails linearity and implies for instance "01 <= 0". lenghth condition |w| = |v| was missing in the second case, corrected

Prop 2.5(3): need hypothesis such as $\emptyset\in\mathcal{F}$? done

Thm 3.2 may be renamed Fact 2.5 since it is proven elsewhere we cite the result and erased the theorem

p.5 l.2: open bracket without closing counterpart done

p.5 ll.9+10 only repeat Def 3.3: omit This was an error, correction and text revision *)

Theorem 3.6 may be be renamed Example 3.6 see **)

p.5 l.-4 and l.-6: "set(B) in cohesive(L)" occurs twice corrected

Proof of Thm 3.8 open ended corrected

Typesetting details (the really few cases where the otherwise impeccable formatting may be reconsidered):

* "lex" appears sometimes in italic bold, sometimes upright bold
* similarly for "min" (p.3 l.8),
* succ, ord, rec_n, word_e, class_k, set.

we used italic bold for functions and upright bold for relations. Since this occurs only for “lex“, we skipped it for the lexicographic ordering

* Using le., th. as abbreviations for Lemma and Theorem abbreviation replaced
* Writing - instead of -- (dash), for instance in l.4 of Remark 5.5 or title of Section 1 done

The presentation is sometimes maybe a bit overly formal, for my taste, and short of motivations of definitions. For instance

* Intro l.6: Why demand that the Qi cover S? The reader might appreciate some motivation here (such as Proposition 2.2) See new text after Definition 2.1.
* Def 3.3: why allow |A'| < |A| ? same thing as *) see new text after def.3.2
* Abstract: What are the main results? abstract has been revised

**) We disagree with the referee at this point. The construction of an unsolvable classification problem without cores is quite general and can be applied to nearly all interesting language families and complexity classes. We revised the following remark to point to this fact. Moreover, the nontrivial proof is in our view too lengthy for an example.

Referee's Report on the paper **"Unsolvability Cores in Classification Problems"**

by Ulricke Brandt and Hermann K.-G Walter

February 22, 2014

General Comments

A *classification problem* is a k-tuple $\mathbf{A} = (A_1, \cdots, A_k)$ of pairwise disjoint infinite subsets $A_i \subseteq S$ for some given set S. Given a class $\mathcal{F} \subseteq 2^S$, a classification problem $\mathbf{A} = (A_1, \cdots, A_k)$ is called $\mathcal{F}$-solvable if there is a partition $\mathbf{Q} = (Q_1, \cdots, Q_k)$ of S (i.e., $S = Q_1 \cup \cdots \cup Q_k$ and $Q_i \cap Q_j = \emptyset$ if $i \neq j$) such that $A_i \subseteq Q_i$ and $Q_i \in \mathcal{F}$ for all $1 \leq i \leq k$. In this case, $\mathbf{Q}$ is a solution of the classification problem $\mathbf{A}$. If $k = 2$, then the corresponding classification problems are called *promise problems*.

Two classification problems $\mathbf{A} = (A_1, \cdots, A_k)$ and $\mathbf{B} = (B_1, \cdots, B_m)$ can be compared in the following way. We say $\mathbf{B} \leq \mathbf{A}$ if $m \leq k$ and there is an injective mapping $\sigma : \{1, \cdots, m\} \to \{1, \cdots, k\}$ such that $B_i \subseteq A_{\sigma(i)}$ for all $1 \leq i \leq m$. (We will call $\mathbf{B}$ a subproblem of $\mathbf{A}$ in this review.) If $\mathcal{F}$ is closed under the union operation, then the $\mathcal{F}$-solvability of $\mathbf{A}$ implies the $\mathcal{F}$-solvability of $\mathbf{B}$ (Proposition 2.2). So, in some sense, this means that $\mathbf{B}$ is "less unsolvable" than $\mathbf{A}$. In other words, the unsolvability of $\mathbf{A}$ seems not necessarily imply the unsociability of $\mathbf{B}$. Surprisingly, there are classification problems so that this implication does hold. These classification problems are called "cores" or "unsolvability cores". More precisely, A classification problem $\mathbf{A}$ is called a k-core of $\mathcal{F}$ if all subproblems of $\mathbf{A}$ is not $\mathcal{F}$-solvable, where $k = |A|$. This paper mainly investigates the existence and their characterizations of cores of different properties.

For example, it shows that (Theorem 3.5), if $\mathcal{F}$ contains $\emptyset$ and S and is closed under the union operation, then a classification problem $\mathbf{A}$, with $|\mathbf{A}| > 1$, is an $\mathcal{F}$-core iff the set $\bigcup \mathbf{A}$ is an $\mathcal{F}$-cohesive sets. Where an infinite set $A \subseteq S$ is called $\mathcal{F}$-cohesive if there is no set Q can split A into two infinite parts (i.e., both $A \cap Q$ and $A \cap Q^c$ are infinite), if both Q and Q^c are in $\mathcal{F}$.

For k=2, the authors proved in a previous paper that, if A is an $\mathcal{F}$-unsolvable classification problem, then there exists a $\mathbf{B} \leq \mathbf{A}$ such that $\mathbf{B}$ is a 2-core of $\mathcal{F}$ as long as $\mathcal{F}$ is closed under union and intersection operations. However, for $k > 2$, this paper proves that, for some $\mathcal{F}$, there is an $\mathcal{F}$-unsolvable classification problem $\mathbf{A}$ such that no $\mathbf{B} \leq \mathbf{A}$ is a core of $\mathcal{F}$ (Theorem 3.6).

The classification problems are extended to the *conditional classification problems* by fixing one of its components. Corresponding results are also proved in the paper. All these are new and theoretically interesting. The proofs are sound. For me, the only drawback of the paper is that it is written in a very technical way with very few explanations. For readers like myself will find it difficult to read first time. But still it is in general a paper of high quality. I recommend to accept the paper for publication in LMCS.

Other Comments

1. Page 1, line 4: If A_i is an infinite subset of S, then S will be infinite automatically. done

2. Page 1, line 5: " such a classification problem is $\mathcal{F}$-solvable, if $\cdots$." Without the prefix it might be confused. done

3. Page 1, line -4: The statement "$\cdots$ we cannot find in general unsolvability cores in unsolvability classification problems with $k > 2$" is not accurate. Theorem 3.6 shows only some special problem $\mathbf{B}$ (i.e., $\mathbf{B} \leq \mathbf{C}(\mathbf{A})$ for some $\mathbf{A}$) that cannot be 3-cores. But Theorem 3.8 does show the existence of k-core (k may be greater than 2) *in general.* see revised text on page 2

4. Page 2, line 1 of section 1: " $\cdots$ we assume that the set family $\mathcal{F} \subseteq \mathbf{2}^S$ "
see revised text on page 2

5. Page 2, line -7: $L_{1,2}$ should be $L_1, L_2 \subseteq X^*$. done

6. Page 3, line -10: "an $\mathcal{F}$-partition" (similar problems in several places)

done, we hope that we found all of them