Cores in Promise and Classification Problems

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CCA Nancy 2013

Promise and Classification Problems - Background

S.Even, A.L.Selman and Y.Yacobi (1985):
promise problems as a generalization of decision problems,
numerous applications

K.Ambos-Spies, U.Brandt, M.Ziegler (2013):
exploration of the impact of constant-size advices on the complexity of classification problems

Subject: solvability and unsolvability of classification problems

Classification Problems

General approach: Basic set S and $k > 1$.

$\mathbf{A} = (A_1, \ldots, A_k)$ is a (straight) k-*classification problem* if and only if

$A_i \subseteq S$, A_i infinite, $A_i \cap A_j = \emptyset$ $(1 \leq i < j \leq k)$.

Special cases :

(1) $k = 2$: *promise problem*.

(2) $\mathbf{A} = (A, B)$ and $B = A^c$ (*complement*) : *decision problem.*

Solvability of Classification Problems

Solvability of classification problems is defined with respect to a set family $\mathcal{F} \subseteq \mathbf{2}^{S}$. For such families we assume, that $\mathcal{F}$ contains all finite and cofinite subsets of S and is closed under finite variation, and associate the families

$\mathcal{F}^{\mathbf{co}} = \{A^{c} \mid A \in \mathcal{F}\}$ (*cofamily*) and $\mathcal{F}^{\mathbf{cc}} = \mathcal{F} \cup \mathcal{F}^{\mathbf{co}}$ (*complement closure*).

Definition : Let $\mathbf{A} = (A_1, \dots, A_k)$ be a k-classification problem and $\mathcal{F} \subseteq \mathbf{2}^{S}$ a set family.

$\mathbf{A}$ is *solvable for* $\mathcal{F} \Leftrightarrow \mathbf{A} \in \boldsymbol{class}_k(\mathcal{F}) \Leftrightarrow$

$\exists\, \mathbf{Q} = (Q_1, \dots, Q_k) : A_i \subseteq Q_i,\ Q_i \in \mathcal{F},\ Q_i \cap Q_j = \emptyset\ (1 \leq i < j \leq k)$ and $Q_1 \cup \dots \cup Q_k = S$

(i.e. $\mathbf{Q}$ is a partition with blocks from $\mathcal{F}$).

Lemma : Let $\mathcal{F}$ be closed under union and intersection. Then

$\mathbf{A} = (A_1, \dots, A_k) \in \boldsymbol{class}_k(\mathcal{F}) \Leftrightarrow (A_i, (A_1 \cup \dots \cup A_k) \setminus A_i) \in \boldsymbol{class}_2(\mathcal{F})\ (1 \leq i \leq k)$

$\Leftrightarrow (A_i, A_j) \in \boldsymbol{class}_2(\mathcal{F})\ (1 \leq i < j \leq k).$

Unsolvability of Classification Problems

We investigate the unsolvability (with respect to $\mathscr{F}$), especially in a strong sense, i.e. unsolvable classification problems such that all subproblems are unsolvable, too - so called **cores** of unsolvability. Cores are defined similarily to complexity cores.

Subjects of our results :

(1) characterization of cores,

(2) existence of cores and

(3) connections to complexity cores.

Applications : language families and complexity classes.

Key to the results : characterization by cohesiveness.

Examples for solvable and unsolvable promise problems

Basic set is $S = X^*$, where X is a finite alphabet. The families used in examples are $\mathscr{L}_{\mathbf{reg}}(X)$ family of regular, $\mathscr{L}_{\mathbf{cf}}(X)$ family of contextfree and $\mathscr{L}_{\mathbf{r.e}}(X)$ family of recursively enumerable languages.

Example :

(1) "*Separation Principle*"

(a) $\exists\, A, B \in \mathscr{L}_{\mathbf{r.e.}}(X) : A \cap B = \emptyset$ and $(A, B) \notin \boldsymbol{class}_2(\mathscr{L}_{\mathbf{r.e.}}(X))$.

(b) $\forall A, B \in \mathscr{L}_{\mathbf{r.e.}}(X)^{\mathbf{co}}, A \cap B = \emptyset : (A, B) \in \boldsymbol{class}_2(\mathscr{L}_{\mathbf{r.e.}}(X))$.

(2) $X = \{a, b\}$.

$A = \{a^n b^n \mid n > 0\}$, $B = \{a^n b^m \mid n, m > 0 \text{ and } n \neq m\}$, $A, B \in \mathscr{L}_{\mathbf{cf}}(X)$

$(A, B) \notin \boldsymbol{class}_2(\mathscr{L}_{\mathbf{reg}}(X))$, $(A, B) \in \boldsymbol{class}_2(\mathscr{L}_{\mathbf{cf}}(X))$.

(Un)solvability and Cohesiveness

Definition : Let $A \subseteq S$, A infinite.

A $\mathscr{F}$- *cohesive* $\Leftrightarrow A \in$ ***cohesive***$(\mathscr{F}) \Leftrightarrow \forall\, Q, Q^c \in \mathscr{F}: A \cap Q$ or $A \cap Q^c$ finite .

Remark : The definition from recursion theory is equivalent to $\mathscr{F}^{\mathbf{cc}}$-cohesiveness.

Lemma : Let $\mathscr{V} \subseteq \mathscr{F}$, where $\mathscr{F}$ is closed under union and variation by $\mathscr{V}$ (i.e. $A \in \mathscr{F}$ and $Q \in \mathscr{V} \Rightarrow A \cap Q, A \cap Q^c \in \mathscr{F}$). Then for all infinite $A, B \in \mathscr{F}$ with $A, B \notin$ ***cohesive***$(\mathscr{V})$ a $Q \in \mathscr{V}$ exists with $(A \cap Q, B \cap Q^c) \in$ ***class***$_2(\mathscr{F})$.

Theorem I: Let (A, B) be a promise problem.

$A \cup B \in$ ***cohesive***$(\mathscr{F}) \Leftrightarrow A, B \in$ ***cohesive***$(\mathscr{F})$ and $(A, B) \notin$ ***class***$_2(\mathscr{F})$.

Cohesive Sets - Known Results

Theorem of Dekker-Myhill : $\mathscr{F}$ denumerable.

$\forall$ A $\subseteq$ S, A infinite $\exists$ A' $\subseteq$ A: A' $\in$ ***cohesive***($\mathscr{F}$).

Theorem of Friedberg :

$\exists$ L $\subseteq$ X*: L $\in \mathscr{L}_{\mathbf{r.e.}}(X)^{\mathbf{co}} \cap$ ***cohesive***($\mathscr{L}_{\mathbf{r.e.}}(X)^{\mathbf{cc}}$).

Cohesive Sets - Additional Results

(1) $\mathscr{L}_{\mathbf{cf}}(X) \cap$ ***cohesive***$(\mathscr{L}_{\mathbf{reg}}(X)) = \emptyset$.

(2) $\forall\ L \in \mathscr{L}_{\mathbf{cf}}(X)\ \exists\ L' \subseteq L$: $L' \in$ ***cohesive***$(\mathscr{L}_{\mathbf{reg}}(X))$ and L' recursive.

(3) $S = \mathbb{N}_0$, $\mathscr{L}_{\mathbf{s\text{-}lin}}$ = family of semilinear subsets of $\mathbb{N}_0$.

(a) $\{2^n \mid n > 0\} \notin$ ***cohesive***$(\mathscr{L}_{\mathbf{s\text{-}lin}})$.

(b) $\{n! \mid n > 0\} \in$ ***cohesive***$(\mathscr{L}_{\mathbf{s\text{-}lin}})$.

Cohesiveness and Immunity

Definition : A infinite.

A is $\mathcal{F}$-*immune* $\Leftrightarrow$ A $\in$ ***immune***($\mathcal{F}$) $\Leftrightarrow$ ($\forall$ B $\in \mathcal{F}$: $B \cap A^c \neq \emptyset$).

Lemma : If $\mathcal{F} = \mathcal{F}^{\mathbf{co}}$ and (A, B) is a promise problem, then

(A, B) $\notin$ ***class***$_2$($\mathcal{F}$) $\Leftrightarrow$ $B^c \in$ ***immune***($\mathcal{F}(A^c)^{\mathbf{co}}$)

($\mathcal{F}(A^c) = \{C \mid C \subseteq A^c \;\&\; C \in \mathcal{F}\}$).

Theorem II : A infinite

A $\in$ ***cohesive***($\mathcal{F}$) $\setminus \mathcal{F} \Rightarrow$ A $\in$ ***immune***($\mathcal{F}$).

Example : X = {a, b}. $A = \{a^n b^n \mid n > 0\}$.

A $\notin$ ***cohesive***($\mathcal{L}_{\mathbf{reg}}$(X)) and A $\in$ ***immune***($\mathcal{L}_{\mathbf{reg}}$(X)).

Cores of Unsolvability

Definition : Let $\mathbf{A} = (A_1, \dots, A_k)$ be a k-classification problem ($k > 1$).

$\mathbf{A}$ is a k-*core of* $\mathscr{F}$ $\Leftrightarrow$ $\mathbf{A} \in \boldsymbol{core}_k(\mathscr{F}) \Leftrightarrow$

For every m-classification problem $\mathbf{B}$, which is a subproblem of $\mathbf{A}$: $\mathbf{B} \notin \boldsymbol{class}_m(\mathscr{F})$.

($\mathbf{B} = (B_1, \dots, B_m)$ m-classification problem, $1 < m \leq k$: $\mathbf{B}$ is a *subproblem* of $\mathbf{A}$ $\Leftrightarrow$

$\exists\, 1 \leq i_1 < \dots < i_m \leq k$: $B_j \subseteq A_{i_j}$ $(1 \leq j \leq m)$)

Cores of Unsolvability and Cohesive sets

Theorem III : Let (A, B) be a promise problem.

$$(A, B) \in \boldsymbol{core}_2(\mathscr{F}) \Leftrightarrow A \cup B \in \boldsymbol{cohesive}(\mathscr{F}).$$

Theorem IV :

If $\mathscr{F}$ is closed under union and $\mathbf{A} = (A_1, \ldots, A_k)$ a k-classification problem $(k > 1)$, then

$$\mathbf{A} \in \boldsymbol{core}_k(\mathscr{F}) \Leftrightarrow A_1 \cup \ldots \cup A_k \in \boldsymbol{cohesive}(\mathscr{F}).$$

Existence of Cores in Promise Problems

Theorem V : Let $\mathscr{F}$ be denumerable and closed under union and intersection.

If (A, B) $\notin$ $\textit{class}_2(\mathscr{F})$ then a subproblem (A', B') of (A, B) exists with

$$(A', B') \in \textit{core}_2(\mathscr{F}).$$

Dekker-Myhill-type "construction": Given $\mathbf{e}_{\mathscr{F}}: \mathbb{N}_0 \to \mathbf{2}^S$ with $\mathbf{e}_{\mathscr{F}}(\mathbb{N}_0) = \mathscr{F}$.

$$(A_0, B_0) := (A, B)\ (\notin \textit{class}_2(\mathscr{F}))$$

$(A_{n+1}, B_{n+1}) :=$ **if** $(A_n \cap \mathbf{e}_{\mathscr{F}}(n), B_n \cap \mathbf{e}_{\mathscr{F}}(n)) \notin \textit{class}_2(\mathscr{F})$

then $(A_n \cap \mathbf{e}_{\mathscr{F}}(n), B_n \cap \mathbf{e}_{\mathscr{F}}(n))$ **else** $(A_n \cap \mathbf{e}_{\mathscr{F}}(n)^c, B_n \cap \mathbf{e}_{\mathscr{F}}(n)^c)$ **fi**

Fact : There exists $g : \mathbb{N}_0 \to \mathbb{N}_0$ with

$$A_{g(n+1)} \subset A_{g(n)},\ A_{g(n)} \subseteq A_n,\ B_{g(n+1)} \subset B_{g(n)} \text{ and } B_{g(n)} \subseteq B_n\ (n \geq 0)$$

Choose $a_n \in A_{g(n+1)} \setminus A_{g(n)}$ and $b_n \in B_{g(n+1)} \setminus B_{g(n)}$ $(n \geq 0)$. Then

$$A' = \{a_n \mid n \geq 0\},\ B' = \{b_n \mid n \geq 0\}.$$

Basic Lemma : If (A, B) $\notin$ $\textit{class}_2(\mathscr{F})$, then for all $Q, Q^c \in \mathscr{F}$:

$(A \cap Q, B \cap Q) \notin \textit{class}_2(\mathscr{F})$ **or** $(A \cap Q^c, B \cap Q^c) \notin \textit{class}_2(\mathscr{F})$.

The closure properties are necessary for th V.

Example : Let $X = \{a, b, c\}$. $|w|_x$ = number of occurences of x in w.

For $x \neq y \in X$: $L_{x,y} = \{w \in X^* \mid |w|_x \neq |w|_y\}$. Then $L_{x,y}$, $L_{x,y}{}^{c} \in \mathscr{L}_{\mathbf{cf}}(X)$.

Consider $L = L_{a,b} \cup L_{b,c} \cup L_{c,a}$ and $L^{c} = \{w \in X^* \mid |w|_a = |w|_b = |w|_c\} = L_{a,b}{}^{c} \cap L_{b,c}{}^{c} \cap L_{c,a}{}^{c}$.

Observe $L \in \mathscr{L}_{\mathbf{cf}}(X)$ and $L^{c} \in \mathscr{L}_{\mathbf{cf}}(X)^{\mathbf{co}} \setminus \mathscr{L}_{\mathbf{cf}}(X)$.

Then the promise problem $(L, L^{c}) \notin$ ***class***$_2(\mathscr{L}_{\mathbf{cf}}(X))$, but contains

no core

with respect to $\mathscr{L}_{\mathbf{cf}}(X)$.

“k = 2” is necessary for th.V

Example : (M.Ziegler). Let X = {a, b, c}.

Consider $\mathscr{L} \subseteq \mathbf{2}^{X^*}$, closed under setting and erasing of leftmarkers and union.

For $A \subseteq X^*$ with $A \notin \mathscr{L}$ or $A^{\mathbf{c}} \notin \mathscr{L}$ define

$$\mathbf{A}(A) = (A_a, A_b, A_c) = (aA \cup bA^{\mathbf{c}}, bA \cup cA^{\mathbf{c}}, cA \cup aA^{\mathbf{c}}).$$

Then $\mathbf{A}(A) \notin \boldsymbol{class}_3(\mathscr{L})$ and for all 3-classification problems **A'**, which are subproblems of $\mathbf{A}(A)$: $\mathbf{A}' \notin \boldsymbol{core}_3(\mathscr{L})$.

Connection to complexity cores - "semicores"

Definition : Let A, B ⊆ S with B infinite.

B is a $\mathscr{F}$*-core of* A ⇔ B ∈ ***core***(A, $\mathscr{F}$) ⇔ ∀ B' ⊆ B, B' infinite: (A, B') ∉ $\boldsymbol{class}_2(\mathscr{F})$.

Corollary : If $\mathscr{F}$ is denumerable and closed under union and intersection, then for all infinite A with A or A^c ∉ $\mathscr{F}$: ***core***(A, $\mathscr{F}$) ≠ Ø.

Connection to complexity cores - Hardcores (Book-Du 1987)

Definition :

B ⊆ A is a *proper $\mathcal{F}$-hardcore* of A if and only if B is infinite and for all C ∈ $\mathcal{F}$, C ⊆ A: B ∩ C is finite.

Theorem : (Book-Du) Let $\mathcal{F}$ be denumerable. A proper $\mathcal{F}$-hardcore of A exists if and only if A is not a finite union of elements of $\mathcal{F}$(A) with a finite set.

Lemma : If $\mathcal{F} = \mathcal{F}^{co}$, A ∩ B = Ø and B infinite then B is a $\mathcal{F}$-core of A if and only if B is a proper $\mathcal{F}$-hardcore of A^c.

Connection to complexity cores - Application

Theorem : (Book-Du) If $\mathscr{L} \subseteq \mathbf{2}^{X^*}$ is recursive and closed under union then for all recursive $A \subseteq X^*$ with $A \notin \mathscr{L}$ a recursive proper hardcore B of A exists.

Theorem VI : If $\mathscr{L} \subseteq \mathbf{2}^{X^*}$ is recursive and closed under boolean operations, then for all recursive $A \subseteq X^*$ with $A^c \notin \mathscr{L}$ a recursive $\mathscr{L}$-core B of A exists.

$\mathscr{L} \subseteq \mathbf{2}^{X^*}$ is called *recursive* if and only if a recursively enumerable representation of $\mathscr{L}$ exists, such that the word-problem is uniformly decidable in this representation, or more formally, there exists $\mathbf{e}_{\mathscr{L}} : \mathbb{N}_0 \rightarrow \mathbf{2}^{X^*}$ such that $\mathbf{e}_{\mathscr{L}}(\mathbb{N}_0) = \mathscr{L}$ and

$$\lambda\, i, j.\ \delta(i, j) = \textbf{if } \boldsymbol{lex}_X(j) \in \mathbf{e}_{\mathscr{L}}(i) \textbf{ then } 1 \textbf{ else } 0 \textbf{ fi}$$

is recursive, where $\boldsymbol{lex}_X$ is the *lexical* enumeration of X^*.