Appendix A

Looking at variation with $\mathscr{L}_{\mathbf{reg}}(X)$ we can weaken the condition "$\mathscr{L} \pm \mathscr{L}_{\mathbf{reg}}(X) \subseteq \mathscr{L}$" substituting $\mathscr{L}_{\mathbf{reg}}(X)$ by a much smaller family at all places where these condition appear. Moreover, we get a much stronger result than le.3.2 for this family. For a given X consider the families $\mathscr{L}_{\mathbf{o}}(X) = (\boldsymbol{fin}(X^*)^{\mathbf{c}})^{\mathbf{ltr}}$ and $\mathscr{L}_{\mathbf{ltr}}(X) = \mathscr{L}_{\mathbf{o}}(X)^{\mathbf{b}}$. Then we can use $\mathscr{L}_{\mathbf{ltr}}(X)$ instead of $\mathscr{L}_{\mathbf{reg}}(X)$. In this appendix we take a more concise look at $\mathscr{L}_{\mathbf{ltr}}(X)$ and derive a complete characterization of ***cohesive***($\mathscr{L}_{\mathbf{ltr}}(X)$). The key to all our considerations are the complement formulas for left translation ($L \subseteq X^*$, $w \in L$) :

(C1) $(wL)^{\mathbf{c}} = wL^{\mathbf{c}} \cup (wX^*)^{\mathbf{c}}$,

(C2) $wL^{\mathbf{c}} = (wL)^{\mathbf{c}} \cap wX^*$.

(C3) $(wX^*)^{\mathbf{c}} = (X^{|w|}X^*)^{\mathbf{c}} \cup (X^{|w|} \setminus w)X^*$ and

(C4) If $L \in \boldsymbol{fin}(X^*)$ and $k > \boldsymbol{max}(|z| \mid z \in L\}$, then $L^{\mathbf{c}} = ((X^kX^*)^{\mathbf{c}} \setminus L) \cup X^kX^*$.

Remark : If $X = \{a\}$, then for all $k \geq 0$: $X^k \setminus a^k = \emptyset$, hence $(a^ka^*)^{\mathbf{c}} \in \boldsymbol{fin}(a^*)$ and $(a^kL)^{\mathbf{c}} \in \boldsymbol{fin}(a^*)^{\mathbf{c}}$ for all $L \in \boldsymbol{fin}(a^*)^{\mathbf{c}}$. This shows $\mathscr{L}_{\mathbf{o}}(a) = \boldsymbol{fin}(a^*)^{\mathbf{c}} = \boldsymbol{fin}(a^*)^{\mathbf{b}}$, i.e. $\mathscr{L}_{\mathbf{ltr}}(a) = \boldsymbol{fin}(a^*)^{\mathbf{b}}$.

Now we prove

Lemma A.1 :

(1) $\mathscr{L}_{\mathbf{o}}(X)^{\mathbf{c}} \subseteq \mathscr{L}_{\mathbf{o}}(X)^{\mathbf{u}}$.

(2) $(\mathscr{L}_{\mathbf{o}}(X)^{\mathbf{u}})^{\mathbf{s}} = \mathscr{L}_{\mathbf{ltr}}(X)$.

Proof : (1) Consider $w \in X^*$ and $L \in \boldsymbol{fin}(X^*)^{\mathbf{c}}$. Then $wL^{\mathbf{c}} \in \mathscr{L}_{\mathbf{o}}(X)$. Since $(X^{|w|}X^*)^{\mathbf{c}} \in \boldsymbol{fin}(X^*)$ and $(X^{|w|} \setminus w)X^* \in \mathscr{L}_{\mathbf{o}}(X)^{\mathbf{u}}$, $(wX^*)^{\mathbf{c}} \in \mathscr{L}_{\mathbf{o}}(X)^{\mathbf{u}}$, $(wL)^{\mathbf{c}} \in \mathscr{L}_{\mathbf{o}}(X)^{\mathbf{u}}$ by (C1) and (C3).

(2) follows by (1).

Lemma A.2 :

(1) $\forall\, A, B \in \boldsymbol{fin}(X^*)$, $u \in X^*$: $uA^{\mathbf{c}} \cap B^{\mathbf{c}} \in \mathscr{L}_{\mathbf{o}}(X)$.

(2) $\forall\, A, B \in \mathscr{L}_{\mathbf{o}}(X)$: $A \cap B \in \mathscr{L}_{\mathbf{o}}(X)$.

Proof : (1) We apply (C4), where additionally $k \geq |u|$. Then

$$B^{\mathbf{c}} \cap uX^* = (((X^kX^*)^{\mathbf{c}} \setminus L) \cap uX^*) \cup (X^kX^* \cap uX^*).$$

Now, $((X^kX^*)^{\mathbf{c}} \setminus L) \cap uX^* = uC$ with $C \in \boldsymbol{fin}(X^*)$ and $(X^kX^* \cap uX^*) = uX^{k-|u|}X^*$. But then

$$uA^{\mathbf{c}} \cap B^{\mathbf{c}} = uA^{\mathbf{c}} \cap (B^{\mathbf{c}} \cap uX^*) = uA^{\mathbf{c}} \cap (uC \cup uX^{k-|u|}X^*) = u(A^{\mathbf{c}} \cap (C \cup X^{k-|u|}X^*)).$$

Since $\boldsymbol{fin}(X^*)^{\mathbf{c}} = (\boldsymbol{fin}(X^*)^{\mathbf{c}})^{\mathbf{b}}$, $A^{\mathbf{c}} \cap (C \cup X^{k-|u|}X^*) \in \boldsymbol{fin}(X^*)^{\mathbf{c}}$ and therefore $uA^{\mathbf{c}} \cap B^{\mathbf{c}} \in \mathscr{L}_{\mathbf{o}}(X)$.

(2) Let $A = uA_1$ and $B = vB_1$ with $A_1, B_1 \in \boldsymbol{fin}(X^*)^{\mathbf{c}}$ and $u,v \in X^*$. If neither $u \leq v$(**pref**) nor $v \leq u$(**pref**), then $A \cap B = \emptyset$. Moreover, if A_1 or $B_1 \in \boldsymbol{fin}(X^*)$, then $A \cap B \in \boldsymbol{fin}(X^*)$. Hence we can assume without loss of generality, that $A_1 = A_2^{\mathbf{c}}$, $B_1 = B_2^{\mathbf{c}}$, $A_2, B_2 \in \boldsymbol{fin}(X^*)$ and $u = vu'$

for some u' ∈ X*. Then by (1) $uA_2^c \cap vB_2^c = v(u'A_2^c \cap B_2^c) \in \mathscr{L}_o(X)$.

Theorem A.3 : $\mathscr{L}_o(X)^{\mathbf{u}} = \mathscr{L}_{\mathbf{ltr}}(X)$ and $\mathscr{L}_{\mathbf{tr}}(X) = \mathscr{L}_{\mathbf{ltr}}(X)^{\mathbf{ltr}}$.

Proof : By le.A.1 $(\mathscr{L}_o(X)^{\mathbf{u}})^{\mathbf{s}} = \mathscr{L}_{\mathbf{ltr}}(X)$. By le.A.2 $(\mathscr{L}_o(X)^{\mathbf{u}})^{\mathbf{s}} = (\mathscr{L}_o(X)^{\mathbf{s}})^{\mathbf{u}} = \mathscr{L}_o(X)^{\mathbf{u}}$.

Furthermore,

$\mathscr{L}_{\mathbf{ltr}}(X) = \mathscr{L}_o(X)^{\mathbf{u}} = ((\mathit{fin}(X^*)^{\mathbf{c}})^{\mathbf{ltr}})^{\mathbf{u}} = (((\mathit{fin}(X^*)^{\mathbf{c}})^{\mathbf{ltr}})^{\mathbf{u}})^{\mathbf{ltr}} = (\mathscr{L}_o(X)^{\mathbf{u}})^{\mathbf{ltr}} = \mathscr{L}_{\mathbf{ltr}}(X)^{\mathbf{ltr}}$.

Next we study ltr-cancellation.

Lemma A.4 : If $\mathscr{L}$ is ltr-cancellative, then $\mathscr{L}^{\mathbf{ltr}}$ and $\mathscr{L}^{\mathbf{u}}$ are ltr-cancellative. If additionally $\mathscr{L} \pm \mathscr{L}_{\mathbf{ltr}}(X) \subseteq \mathscr{L}$, then $\mathscr{L}^{\mathbf{co}}$, $\mathscr{L}^{\mathbf{s}}$ and $\mathscr{L}^{\mathbf{b}}$ are ltr-cancellative.

Proof : (1) Suppose $wL \in \mathscr{L}^{\mathbf{ltr}}$, then wL = uL' for w, u ∈ X* and L' ∈ $\mathscr{L}$. Then either $w \le u$(**pref**) or $u \le w$(**pref**). If u = wv, then wL = w(vL) and therefore L = vL'. Since L' ∈ $\mathscr{L}$, L ∈ $\mathscr{L}^{\mathbf{ltr}}$. If w = uv, then vL = L' ∈ $\mathscr{L}$. $\mathscr{L}$ is ltr-cancellative, hence L ∈ $\mathscr{L}$.

(2) Let $wL = L_1 \cup ... \cup L_n$ with $L_i \in \mathscr{L}$ for $1 \le i \le n$. Then each $L_i \subseteq wX^*$, i.e. $L_i = wL_i'$. Since $\mathscr{L}$ is ltr-cancellative, $L_i' \in \mathscr{L}$. But then $L = L_1' \cup ... \cup L_n' \in \mathscr{L}^{\mathbf{u}}$.

(3) If $wL \in \mathscr{L}^{\mathbf{co}}$, then $(wL)^c \in \mathscr{L}$. But $(wL)^c \cap wX^* = wL^c \in \mathscr{L}$. Since $\mathscr{L}$ is ltr- cancellative, $L^c \in \mathscr{L}$, hence $L \in \mathscr{L}^{\mathbf{co}}$.

(4) By fact 1.2.(2) $\mathscr{L}^{\mathbf{co}} \pm \mathscr{L}_{\mathbf{ltr}}(X) \subseteq \mathscr{L}^{\mathbf{co}}$ and $\mathscr{L}^{\mathbf{u}} \pm \mathscr{L}_{\mathbf{ltr}}(X) \subseteq \mathscr{L}^{\mathbf{u}}$. $\mathscr{L}^{\mathbf{co}}$ and $\mathscr{L}^{\mathbf{u}}$ are ltr-cancellative. Moreover, $\mathscr{L}^{\mathbf{s}} = ((\mathscr{L}^{\mathbf{co}})^{\mathbf{u}})^{\mathbf{co}}$ and $\mathscr{L}^{\mathbf{b}} = ((\mathscr{L}^{\mathbf{c}})^{\mathbf{u}})^{\mathbf{s}}$, hence $\mathscr{L}^{\mathbf{s}}$ and $\mathscr{L}^{\mathbf{b}}$ are ltr-cancellative.

Theorem A.5 : If #(X) > 1, then $\mathscr{L}_{\mathbf{ltr}}(X)$ is ltr-cancellative.

Proof : Since $((\mathit{fin}(X^*)^{\mathbf{c}})^{\mathbf{ltr}})^{\mathbf{u}} = \mathscr{L}_{\mathbf{ltr}}(X)$, we can apply le.A.4 , if $\mathit{fin}(X^*)^{\mathbf{c}}$ is ltr-cancellative. Let $wL \in \mathit{fin}(X^*)^{\mathbf{c}}$. If $wL \in \mathit{fin}(X^*)$, then $L \in \mathit{fin}(X^*)$. Suppose $wL \in \mathit{fin}(X^*)^{\mathbf{c}}$, then by (C1) $(wL)^c = wL^c \cup (wX^*)^c \in \mathit{fin}(X^*)$. But $(wX^*)^c \notin \mathit{fin}(X^*)$ and we arrive at a contradiction to $(wL)^c \in \mathit{fin}(X^*)$, unless w = **1**. If w = **1**, we get directly $L = wL \in \mathit{fin}(X^*)^{\mathbf{c}}$.

As indicated, we can determine ***cohesive***$(\mathscr{L}_{\mathbf{ltr}}(X))$ using sequential mappings.

Theorem A.6 : Let #(X) > 1. A ∈ ***cohesive***$(\mathscr{L}_{\mathbf{ltr}}(X))$ if and only A ∉ $\mathit{fin}(X^*)$ and a sequential $f_A : \mathbb{N}_0 \to X^*$ exists, with $A \setminus f_A(n)X^* \in \mathit{fin}(X^*)$ for all $n \ge 0$.

Proof : The key to the proof is the following

Assertion 1 : If A ∈ ***cohesive***$(\mathscr{L}_{\mathbf{ltr}}(X))$, then

$$\forall\, u, v \in X^*, |u| = |v| : A \cap uX^*, A \cap vX^* \notin \mathit{fin}(X^*) \Rightarrow u = v.$$

Proof : Suppose $A \cap uX^* \notin \mathit{fin}(X^*)$ and $u \ne v$. Then $uX^* \cap vX^* = \emptyset$. Hence, $vX^* \cap A \subseteq (uX^*)^c \cap A$ and therefore $(uX^*)^c \cap A \notin \mathit{fin}(X^*)$. Hence A ∉ ***cohesive***$(\mathscr{L}_{\mathbf{ltr}}(X))$ - a contradiction.

Suppose A ∈ ***cohesive***$(\mathscr{L}_{\mathbf{ltr}}(X))$. Since A ∉ $\mathit{fin}(X^*)$, we can find to any $n \ge 0$ w ∈ X* with

$|w| = n$ and $A \cap wX^* \notin \textit{fin}(X^*)$. Define $f_A(n) = w$. By the assertion f_A is uniquely determined. If $u \le w(\mathbf{pref})$, then $A \cap wX^* \subseteq A \cap uX^*$, hence $A \cap uX^* \notin \textit{fin}(X^*)$ and by assertion $f_A(|u|) = u$. Moreover, since $A \in \textit{cohesive}(\mathscr{L}_{\mathbf{ltr}}(X))$, $A \cap (f_A(n)X^*)^{\mathbf{c}} \in \textit{fin}(X^*)$ for all $n \ge 0$.

Conversely, let $A \notin \textit{fin}(X^*)$ and f_A sequential with $A \setminus f_A(n)X^* \in \textit{fin}(X^*)$ $(n \ge 0)$. Consider wL for $w \in X^*$ and $L \in \textit{fin}(X^*)^{\mathbf{c}}$.

Assertion 2 : $wL \cap A \notin \textit{fin}(X^*) \Leftrightarrow (wL)^{\mathbf{c}} \cap A \in \textit{fin}(X^*)$.

Proof : Suppose $wL \cap A \notin \textit{fin}(X^*)$. If $L \in \textit{fin}(X^*)$, then $wL \cap A \in \textit{fin}(X^*)$ - a contradiction. Assume $L = L'^{\mathbf{c}}$ for $L' \in \textit{fin}(X^*)$. If $f_A(|w|) \ne w$, then $wL'^{\mathbf{c}} \subseteq wX^*$ and $wX^* \cap f(|w|)X^* = \emptyset$. But then $wL'^{\mathbf{c}} \subseteq (f(|w|)X^*)^{\mathbf{c}}$ and $wL \cap A = wL'^{\mathbf{c}} \cap A \in \textit{fin}(X^*)$, again a contradiction. Therefore, $f(|w|) = w$. We know $(wL'^{\mathbf{c}})^{\mathbf{c}} \cap A = (wL' \cap A) \cup ((X^{|w|}X^*)^{\mathbf{c}} \cap A) \cup (X^{|w|} \setminus w)X^* \cap A)$. Since $L' \in \textit{fin}(X^*)$, $wL' \cap A \in \textit{fin}(X^*)$, since $(X^{|w|}X^*)^{\mathbf{c}} \in \textit{fin}(X^*)$, $(X^{|w|}X^*)^{\mathbf{c}} \cap A \in \textit{fin}(X^*)$, too. Moreover, $(X^{|w|} \setminus w)X^* \subseteq (wX^*)^{\mathbf{c}}$ and then $(X^{|w|} \setminus w)X^* \cap A \subseteq (wX^*)^{\mathbf{c}} \cap A = (f(|w|)X^*)^{\mathbf{c}} \cap A \in \textit{fin}(X^*)$. This shows $(wL)^{\mathbf{c}} \cap A \in \textit{fin}(X^*)$.

Let $L = w_1L_1 \cup \ldots \cup w_kL_k \in \mathscr{L}_{\mathbf{ltr}}(X)$ with $w_i \in X^*$ and $L_i \in \textit{fin}(X^*)^{\mathbf{c}}$ for all $1 \le i \le k$. If $L \cap A \notin \textit{fin}(X^*)$, then $1 \le i \le k$ exists with $w_iL_i \cap A \notin \textit{fin}(X^*)$. By the assertion $(w_iL_i)^{\mathbf{c}} \cap A \in \textit{fin}(X^*)$. But then $L^{\mathbf{c}} \cap A = (w_1L_1)^{\mathbf{c}} \cap \ldots \cap (w_kL_k)^{\mathbf{c}} \cap A \in \textit{fin}(X^*)$.

Note, that by th.A.2 $L, L^{\mathbf{c}} \in \mathscr{L}_{\mathbf{ltr}}(X)$. This proves $A \in \textit{cohesive}(\mathscr{L}_{\mathbf{ltr}}(X))$.

The functions f_A are uniquely for $A \in \textit{cohesive}(\mathscr{L}_{\mathbf{ltr}}(X))$. But we can show more.

Corollary : Let $\#(X) > 1$.

(1) If $A \in \textit{cohesive}(\mathscr{L}_{\mathbf{ltr}}(X))$ and $B \subseteq A$ with $B \notin \textit{fin}(X^*)$, then $f_A = f_B$.

(2) If $A, B \in \textit{cohesive}(\mathscr{L}_{\mathbf{ltr}}(X))$, then $A \cup B \in \textit{cohesive}(\mathscr{L}_{\mathbf{ltr}}(X))$ if and only if $f_A = f_B$.

Proof : (1) Suppose $n \ge 0$ exists with $f_A(n) \ne f_B(n)$. We know $B \cap f_B(n)X^*, A \cap f_A(n)X^* \notin \textit{fin}(X^*)$ and $B \cap f_B(n)X^* \subseteq A \cap f_B(n)X^*$. This is a contradiction to ass.1.

(2) If $A \cup B \in \textit{cohesive}(\mathscr{L}_{\mathbf{ltr}}(X))$, then by (1) $f_A = f_{A \cup B} = f_B$, since $A, B \subseteq A \cup B$. Conversely, $(A \cup B) \cap (f_A(n)X^*)^{\mathbf{c}} = (A \cap (f_A(n)X^*)^{\mathbf{c}} \cup (B \cap (f_B(n)X^*)^{\mathbf{c}}) \in \textit{fin}(X^*)$ $(n \ge 0)$.

Since $f_A = f_B$, $A \cup B \in \textit{cohesive}(\mathscr{L}_{\mathbf{ltr}}(X))$ by th.A.6.

Example A.7 : Let X with $\#(X) > 1$. Define $\textit{pref}(L) = \{u \mid \exists\, w \in L: u \le w(\mathbf{pref})\}$ for $L \subseteq X^*$. Let $u, w, v \in X^*$ with $w \ne \mathbf{1}$ and $A = uw^*v$. Define $f_A(n) = z$ with $|z| = n$ and $z \in \textit{pref}(L)$. Then for $n \ge 0$ $A \setminus f_A(n)X^* \in \textit{fin}(X^*)$ and therefore $uw^*v \in \textit{cohesive}(\mathscr{L}_{\mathbf{ltr}}(X))$.

Remark : Everything can be done for rightmarking. Considering *right translation* "Lw" and the closure operation $\mathscr{L}^{\mathbf{rtr}} = \{Lw \mid w \in X^* \text{ and } L \in \mathscr{L}\}$, we obtain $\mathscr{L}_{\mathbf{rtr}}(X) = ((\textit{fin}(X^*)^{\mathbf{c}})^{\mathbf{rtr}})^{\mathbf{u}}$, which is a "rtr-cancellative" boolean algebra.

One can show, that for $X = \{a, b\}$ we get $aX^* \notin \mathscr{L}_{\mathbf{rtr}}(X)$, $X^*b \notin \mathscr{L}_{\mathbf{ltr}}(X)$, $a^* \notin \mathscr{L}_{\mathbf{ltr}}(X) \cap \mathscr{L}_{\mathbf{rtr}}(X)$.

Appendix B

To assert solvability of a promise problem one can use the reduction principle ([8]) in connection with decompositions of sets for a set family.

Definition B.1 : Let A, B ∈ $\mathcal{S}\backslash$***fin***($\mathcal{S}$). (A, B) is *$\mathcal{S}$-reducible* if and only if A' ⊆ A and B' ⊆ B exist with A', B' ∈ $\mathcal{S}\backslash$***fin***($\mathcal{S}$), A' ∪ B' = A ∪ B and A' ∩ B' = Ø.

Lemma B.2 : Let A, B ∈ $\mathcal{S}\backslash$***fin***($\mathcal{S}$) with A ∩ B = Ø. If (A^c, B^c) is $\mathcal{S}^{co}$ -reducible, then (A, B) ∈ ***promise***($\mathcal{S}$).

Proof : Since A ∩ B = Ø, we know $A^c \cup B^c$ = S. Then $A'^c \subseteq A^c$ and $B'^c \subseteq B^c$ exist with A', B' ∈ $\mathcal{S}\backslash$***fin***($\mathcal{S}$), $A'^c \cup B'^c = A^c \cup B^c$ = S and $A'^c \cap B'^c$ = Ø. By construction A ⊆ A', B ⊆ B' and B' = A'^c, i.e. (A, B) ∈ ***promise***($\mathcal{S}$).

Example B.3 : Consider A, B ∈ $\mathcal{L}_{r.e.}(X)^{co}$ with A ∩ B = Ø, A, A^c, B, B^c ∉ ***fin***(X*). we know A^c, B^c ∈ $\mathcal{L}_{r.e}$(X). By a theorem of Friedberg ([8]) (A^c, B^c) is $\mathcal{L}_{r.e}$(X)-reducible. Applying le.B.2 yields (A, B) ∈ ***promise***($\mathcal{L}_{r.e.}(X)^{co}$). In contrast to this result A, B ∈ $\mathcal{L}_{r.e.}$(X) with A ∩ B = Ø exist, such that (A, B) ∉ ***promise***($\mathcal{L}_{r.e.}$(X)) [8].

Example B.4 : Let X = {a, b}. Consider

$$A = \{a^n b^n a^m \mid n, m \geq 0\} \text{ and } B = \{a^m b^n a^n \mid n, m \geq 0\}.$$

Then A, B ∈ $\mathcal{L}_{cf}$(X). Suppose A', B' ∈ $\mathcal{L}_{cf}$(X) \ ***fin***(X*) can be found, such that A ∪ B = A' ∪ B', A' ⊆ A, B' ⊆ B and A' ∩ B' = Ø. Let C = A ∩ B. Then A' ∩ C ∉ ***fin***(X*)) or B' ∩ C ∉ ***fin***(X*). Suppose the first case. Then we can apply Ogden's lemma to $a^n b^n a^n$ ∈ A' for n large enough with the marking $\underline{\mathbf{a}}^n b^n a^n$. But then we find k ≠ n with $a^k b^n a^n$ ∈ A' ⊆ A, which is not possible. We can handle the case "B' ∩ C ∉ ***fin***(X*)" analogously. Hence (A, B) is not $\mathcal{L}_{cf}$ (X)-reducible.

Reducibility of (A, B) can be connected to decomposability of sets C.

Definition B.5 : Let A ∈ $\mathcal{S}\backslash$***fin***($\mathcal{S}$). Then A is *$\mathcal{S}$-decomposable* if and only if A', A" ⊆ A exist with A', B' ∈ $\mathcal{S}\backslash$***fin***($\mathcal{S}$), A' ∪ B' = A and A' ∩ B' = Ø.

Lemma B.6 : Let $\mathcal{S}$, $\mathcal{S}'$ ⊆ $\mathbf{2}^{\mathbf{S}}$ with $\mathcal{S} \odot \mathcal{S}'^{co} \subseteq \mathcal{S}$ and $\mathcal{S} \odot$ ***fin***$(S)^{co} \subseteq \mathcal{S}$.

If A, B ∈ $\mathcal{S}\backslash$***fin***($\mathcal{S}$) such that A ∩ B is $\mathcal{S}'$-decomposable, then (A, B) is $\mathcal{S}$-reducible.

Proof : Consider A ∩ B. Then two cases arise

Case 1 : "A ∩ B ∈ ***fin***(S)" In this case define A' = A and B' = B \ (A ∩ B). Since $\mathcal{S} \odot$ ***fin***$(S)^{co}$ ⊆ $\mathcal{S}$, A', B' ∈ $\mathcal{S}\backslash$***fin***($\mathcal{S}$). Moreover, A' ∩ B' = Ø and A ∪ B = A' ∪ B'.

Case 2 : "A ∩ B ∉ ***fin***(S)" Since A ∩ B is $\mathcal{S}'$-decomposable, we find A_0, B_0 ∈ $\mathcal{S}'\backslash$***fin***(S) with $A_0 \cap B_0$ = Ø and $A_0 \cap B_0$ = A ∩ B. Define A' = A ∩ B_0^c and B' = B ∩ A_0^c. Since $\mathcal{S} \odot \mathcal{S}'^{co}$ ⊆ $\mathcal{S}$, A', B' ∈ $\mathcal{S}\backslash$***fin***($\mathcal{S}$). Moreover, A' ∩ B' = Ø and A ∪ B = A' ∪ B'.

Decomposability is present in many language families and complexity classes.

Fact B.7 : If $\mathscr{L} \subseteq \mathscr{L}_{\mathbf{lreg}}(X)$ with $\mathscr{L} \odot \mathscr{L}_{\mathbf{reg}}(X) \subseteq \mathscr{L}$ and $L \in \mathscr{L} \setminus \boldsymbol{fin}(X^*)$, then L is $\mathscr{L}$-decomposable.

Proof : Since $|L| \in \mathscr{L}_{\mathbf{lreg}}(X) \setminus \boldsymbol{fin}(X^*)$ we find by the pumping lemma $\alpha > 0$ and $\beta \geq 0$ such that $a^{\beta}(a^{\alpha})^* \subseteq |L|$. Consider $R = \lambda_X^{-1}(a^{\beta}(a^{2\alpha})^* \in \mathscr{L}_{\mathbf{reg}}(X)$. Moreover, $A = L \cap R$, $B = L \cap R^{\mathrm{c}} \in \mathscr{L} \setminus \boldsymbol{fin}(X^*)$ (by assumption). But $A \cup B = L$ and $A \cap B = \emptyset$.

Fact B.8 : Let $\mathscr{S} \odot \mathscr{V} \subseteq \mathscr{S}$ and $\mathscr{V} = \mathscr{V}^{\mathbf{c}}$.

If $A \notin \boldsymbol{cohesive}(\mathscr{S}) \cup \boldsymbol{fin}(X^*)$, then A is $\mathscr{S}$- decomposable.

Proof : Since $A \notin \boldsymbol{cohesive}(\mathscr{S}) \cup \boldsymbol{fin}(X^*)$ a $Q \in \mathscr{V}$ exists with $A \cap Q$, $A \cap Q^{\mathrm{c}} \notin \boldsymbol{fin}(X^*)$.