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The Position of Index Sets of Identifiable Sets in the Arithmetical Hierarchy

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We prove that every set of partial recursive functions which can be identified by an inductive inference machine is included in some identifiable function set with index set in $\Sigma_3 \cap \Pi_3$. An identifiable set is presented with index set in $\Sigma_3 \cap \Pi_3$ but neither in Σ_2 nor in Π_2. Furthermore we show that there is no nonempty identifiable set with index set in Σ_1. In Π_1 it is possible to locate this king of set. In the last part of the paper we show that the problem to identify all partial recursive functions and the halting problem are of the same degree of unsolvability.

INTRODUCTION

Gold (1967), shows that there exists no inductive inference machine identifying the set **R** of all recursive functions, and similarly there is none to identify the set **P** of all partial recursive (p.r.) functions. The question is how to characterize identifiable subsets of **P**. In this paper identifiable function sets are studied in terms of their "complexity," where the complexity of a set M of p.r. functions is measured by the position of the corresponding index set in Kleene's arithmetical hierarchy. We shall show that every identifiable set is included in an identifiable set with index set in $\Sigma_3 \cap \Pi_3$ and that there is no nonempty identifiable set with index set in Σ_1. It is easily shown that there is an identifiable set with index set in Π_1. Moreover we exhibit an identifiable set with index set in $\Sigma_3 \cap \Pi_3$ but neither in Π_2 nor in Σ_2. With respect to the topic also compare with Klette (1976). He investigates identifiable subsets of **R** and shows that the corresponding index sets are in Σ_3 and that there exists an identifiable subset of **R** with Σ_3-complete index set.

The paper is completed by the answer to the problem of how complex an inductive inference machine must be to identify all partial recursive functions. Clearly, these inference machines have to use oracles. Adleman and Blum (1975) show that the problem to identify all recursive functions

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is strictly easier than the halting problem. We show that the problem to identify all partial recursive functions and the halting problem are of equivalent degree of unsolvability.

Basic Notations and Definitions

We assume that the reader is familiar with the basic concepts and results of recursion theory (Rogers, 1967). We adopt the notations of (Rogers, 1967); in particular an acceptable enumeration of the unary p.r. functions is denoted by $(\phi_i)_{i=0}^{\infty}$. For every p.r. function f we define the index set of f to be the set $\text{Ind}(f) = \{i/\phi_i = f\}$. If M is a set of p.r. functions then $\text{Ind}(M) = \{i/\phi_i \in M\}$ is the index set of M.

An inductive inference machine (INM) is an algorithmic device working as follows: The machine starts in some initial state with blank memory. From there, it proceeds autonomously except that, from time to time, the device requests an input or produces an output. Possible inputs are pairs of natural numbers (x, y) or *, while outputs are natural numbers (see Fig. 1).

We demand that an INM requests during every computation infinitely many inputs and that it outputs an infinite sequence of natural numbers.

Next, we define which partial recursive functions a given INM can identify. For this the following terminology introduced by (Blum and Blum, 1975) is useful: For any p.r. function f we say that **f is an enumeration** of f if and only $\mathbf{f} = \langle a_0, a_1, ..., \rangle$ is an infinite sequence where:

(i) $a_i \in \mathbf{N}^2 \cup \{*\}$ for every $i \in \mathbf{N}$ and

(ii) $f(x) = y \Leftrightarrow \exists i \in \mathbf{N}: a_i = (x, y)$.

If m is an INM, then m **converges to** z **under input** $\mathbf{f} = \langle a_0, a_1, ..., \rangle$ ($m[\mathbf{f}]\downarrow z$) if and only if m produces with input $\langle a_0, a_1, ..., \rangle$ an output sequence $\langle i_0, i_1, ... \rangle$ converging to z, that is: $\exists y \in N\, \forall x \geqslant y\, (i_x = z)$.

Otherwise we write $m[\mathbf{f}]\uparrow$ and say that m **under input f diverges.** $m[\mathbf{f}]\downarrow$ means that m converges to some natural number.

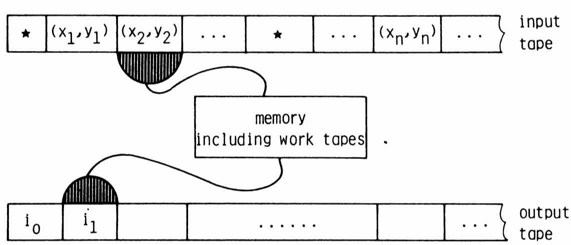

FIGURE 1

m **identifies** a p.r. function f if and only if for every enumeration $\mathbf{f}$ of f there is a z such that $m[\mathbf{f}]\downarrow z$ and ϕ_z is an extension of f ($f \subseteq \phi_z$), i.e., $\phi_z(x) = f(x)$ for every $x \in \mathrm{Dom}(f)$.

By M_m we denote the set of all p.r. functions which the INM m identifies. A set M of p.r. functions is called **identifiable** if and only if there is an INM m with $M_m = M$.

Now, we introduce a special kind of INMs, namely the sharp INMs. An INM m is called **sharp** if and only if for every p.r. $f \notin M_m$ either $m[\mathbf{f}]\uparrow$ for every enumeration $\mathbf{f}$ of f or for every enumeration $\mathbf{f}$ of f there is an z such that $m[\mathbf{f}]\downarrow z$ and $f \not\subseteq \phi_z$.

Blum and Blum (1975) show (see Theorem 2) that for every INM m there is a sharp INM m' with $M_m \subseteq M_{m'}$. Therefore we are particularly interested in sharp INMs.

For every sharp INMm

$$K_m = \{f \in \mathbf{P}/f \notin M_m \wedge \forall \mathbf{f} \quad \text{enumeration of} \quad f\colon m[\mathbf{f}]\downarrow\}$$

and

$$D_m = \{f \in \mathbf{P}/f \notin M_m \wedge \forall \mathbf{f} \quad \text{enumeration of} \quad f\colon m[\mathbf{f}]\uparrow\}.$$

Observe that every sharp INM m the sets M_m, K_m, and D_m define a partition of $\mathbf{P}$.

In the set of sharp INMs we are particularly interested in so-called "strong" and "weak" INMs. A sharp INMm is called **strong** if $K_m = \phi$ and **weak** if $D_m = \phi$. Thus a strong INM diverges for all input sequences of functions not inferred whereas a weak INM converges to indices of functions which are not extensions of the given function. (In the literature the term "reliable" is also used instead of "strong.") A set M of p.r. functions is called **sharply (strongly, weakly) identifiable** if and only if there is a sharp (strong, weak) INMm with $M_m = M$.

An upper bound

In this section we use the classification of index sets in the $\Sigma_{m,n}$ $\Pi_{m,n}$-hierarchy of (Hay, 1974). First we show that the index set of every sharply identifiable set of p.r. functions is in $\Sigma_{2,2} = \Pi_{2,2}$, i.e., it is a difference of Σ_2 sets. To do this we define the limits of functions following the notation of Gold, (1965). Given a recursive function $g\colon \mathbf{N}^2 \to \mathbf{N}$ and an $i \in \mathbf{N}$, $\lim_x g(i, x) = z$ will signify that there is an y such that, for all $x > y$, $g(i, x) = z$. In the case that $\forall y \exists x > y(g(i, y) \neq g(i, x))$ we say that $\lim_x g(i, x)$ is undefined.

THEOREM 1. (a) *For every sharp INMm* $\mathrm{Ind}(M_m) \in \Sigma_{2,2} = \Pi_{2,2}$.

(b) *If M is strongly identifiable then* $\mathrm{Ind}(M) \in \Sigma_2$ *and if M is weakly identifiable then* $\mathrm{Ind}(M) \in \Pi_2$.

Proof. $\text{Ind}(M_m) = \overline{\text{Ind}(D_m)} - \text{Ind}(K_m)$ since $\overline{D_m \cup K_m} = M_m$. We show $\text{Ind}(D_m) \in \Pi_2$ and $\text{Ind}(K_m) \in \Sigma_2$.

Consider a recursive function $D\colon \mathbf{N}^2 \to \mathbf{N}^2 \cup \{*\}$ such that for every i $\langle D(i, 0), D(i, 1),..., \rangle$ is an enumeration of ϕ_i. Define the recursive function g by $g(i, x) =$ "xth output of m under the enumeration $\langle D(i, 0), D(i, 1),... \rangle$ of ϕ_i."

Since m is sharp we get

$$\text{Ind}(M_m) = \{i/\exists z \lim_x g(i, x) = z \text{ and } \phi_i \subseteq \phi_z\},$$

$$\text{Ind}(K_m) = \{i/\exists z \lim_x g(i, x) = z \text{ and } \phi_i \nsubseteq \phi_z\},$$

and

$$\text{Ind}(D_m) = \{i/\lim_x g(i, x) \text{ is undefined}\}.$$

Obviously $\text{Ind}(D_m) \in \Pi_2$, since

$$\text{Ind}(D_m) = \{i/\forall x \exists y (y > x \wedge g(i, x) \neq g(i, y))\}.$$

To see that $\text{Ind}(K_m) \in \Sigma_2$ observe that

$$\begin{aligned}\text{Ind}(K_m) &= \left\{i/\exists z \left(\lim_x g(i, x) = z \wedge \phi_i \nsubseteq \phi_z\right)\right\} \\ &= \{i/\exists z \exists y (\forall x > y (g(i, x) = z) \\ &\quad \wedge \exists u (u \in \text{Dom}(\phi_i) \\ &\quad \wedge (u \notin \text{Dom}(\phi_z) \vee \phi_i(u) \neq \phi_z(u))))\} \\ &= \{i/\exists z \exists y \exists u \forall x ((x > y \Rightarrow g(i, x) = z) \\ &\quad \wedge \phi_i(u)\downarrow \wedge (\phi_z(u)\uparrow \vee \phi_i(u) \neq \phi_z(u)))\}\end{aligned}$$

which is in Σ_2 by the Tarski–Kuratowski algorithm. ∎

COROLLARY. *If M is sharply identifiable then* $\text{Ind}(M) \in \Sigma_3 \cap \Pi_3$.

By the next theorem we show that the upper bound given in the corollary is sharp in the sense that there exists a sharply identifiable function set with index set in $\Sigma_3 \cap \Pi_3$ but neither in Σ_2 nor in Π_2. To do this we introduce the following sets L and S of partial recursive functions. Let $F = \{f \in \mathbf{R}/\forall x(f(x) \neq 1)$ and $\exists z(f(x) = z$ almost everywhere$)\}$ then $L = \{g \in \mathbf{P}/\exists f \in F(g \subseteq f)\}$. Hence L contains every p.r. function which can be extended to some almost everywhere constant recursive function never adopting 1 as its value. Let S be the set of all selfdescribing p.r. functions that means $S = \{f \in \mathbf{P} / \exists z(f(z) = 1)$ and

$$\forall x(f(x) = 1 \wedge \forall y(y < x \Rightarrow f(y) \neq 1) \Rightarrow f \subseteq \phi_x)\}.$$

For every function in S the minimal x with $f(x) = 1$ is an index of an extension of f. Now we can show that $S \cup L$ has just the announced properties.

THEOREM 2. *$S \cup L$ is sharply identifiable and* $\mathrm{Ind}(S \cup L) \in (\Sigma_3 \cap \Pi_3) - (\Sigma_2 \cup \Pi_2)$.

Proof. There exists a sharp INMm with $M_m = S \cup L$. Let $(f_i)_{i=0}^{\infty}$ be an effective enumeration of the total functions in L.

Then m works as follows: it requests inputs and conjectures the least x, if any, such that $(x, 1)$ occurs under the inputs received so far. If no such x exists m outputs after requesting the nth input a_n, the least i such that for every $j \leqslant n$ with $a_j = (x_j, y_j)$, $f_i(x_j) = y_j$. m has the desired properties.

It converges for every $\mathbf{f}$ with $f \in S \cup L$ to an i with $f \subseteq \phi_i$ and for every $\mathbf{f}$ with $1 \in f(\mathbf{N})$ but $f \notin S$ to an i with $f \nsubseteq \phi_i$. m diverges for every $\mathbf{f}$ of an f with $l \notin f(\mathbf{N})$ which changes its value infinitely often. That $\mathrm{Ind}(S \cup L) \in \Sigma_3 \cap \Pi_3$ follows directly by the corollary to Theorem 1.

It remains to prove $\mathrm{Ind}(S \cup L) \notin \Sigma_2 \cup \Pi_2$. Let $\mathrm{FIN} = \{i / W_i \text{ is finite}\}$, $\mathrm{INF} = \overline{\mathrm{FIN}}$. It suffices to show that FIN, $\mathrm{INF} \leqslant_m \mathrm{Ind}(S \cup L)$. Define

$$\phi_{r(i)}(x) = \begin{cases} x+2 & \text{if } \quad |W_i| > x, \\ \uparrow & \text{otherwise.} \end{cases}$$

Then $i \in \mathrm{FIN} \Leftrightarrow r(i) \in \mathrm{Ind}(L) \Leftrightarrow r(i) \in \mathrm{Ind}(S \cup L)$, since $l \notin \text{range } \phi_{r(i)}$; hence $\mathrm{Ind}(S \cup L) \notin \Pi_2$. To show $\mathrm{INF} \leqslant_m \mathrm{Ind}(S \cup L)$, define a recursive function f by

$$\phi_{f(m,n)}(x) = \begin{cases} 1 & \text{if } \quad x = n \\ \phi_m(x) & \text{otherwise.} \end{cases}$$

According to the recursion theorem, for every m, an l can be effectively found with $\phi_{f(m,l)} = \phi_l$. Let g be the recursive function assigning to every m this l. Define a recursive $h(i, j)$ by

$$\phi_{h(i,j)}(x) = \begin{cases} \uparrow & \text{if } \quad x < g(j), \\ 1 & \text{if } \quad x = g(j), \\ \phi_i(x) & \text{otherwise.} \end{cases}$$

Clearly, $\phi_{h(i,j)}(x)=\phi_i(x)$ for almost all x. Define, for every recursive function t, $\text{INC}_t=\{(i,j)/\forall x>t(j)(\phi_i(x)\downarrow\ \Rightarrow\phi_i(x)=\phi_j(x))\}$. Then $(i,j)\in \text{INC}_g \Leftrightarrow \phi_{h(i,j)}\subseteq\phi_{g(j)}\Leftrightarrow h(i,j)\in \text{Ind}(S)$. Finally, let

$$\phi_{s(i)}(x)=\begin{cases} x & \text{if } \ |W_i|>x \text{ or } x=1,\\ \uparrow & \text{otherwise,}\end{cases}$$

and $\phi_k(x)=x$ for all x. Then

$$\begin{aligned} i\in \text{INF} &\Leftrightarrow \phi_{s(i)}=\phi_k\\ &\Leftrightarrow (k,s(i))\in \text{INC}_g\\ &\Leftrightarrow h(k,s(i))\in \text{Ind}(S)\\ &\Leftrightarrow h(k,s(i))\in \text{Ind}(SUL)\end{aligned}$$

since $1\in$ range $\phi_{h(k,s(i))}$; hence $\text{Ind}(SUL)\notin\Sigma_2$. ▮

By Theorem 1 $\text{Ind}(SUL)\in\Sigma_{2,2}$. On the other hand $\overline{\text{Ind}(SUL)}=\overline{\text{Ind}(S)}-\text{Ind}(L)\in\Sigma_{2,2}$ as well, since

$$\begin{aligned}\text{Ind}(L)=\{i/\forall y(\phi_i(y)\neq 1)\\ \wedge\ \exists x,z\,\forall y(y\geqslant x\Rightarrow(\phi_i(y)\downarrow\ \Rightarrow\phi_i(y)=z))\}\in\Sigma_2\end{aligned}$$

and

$$\begin{aligned}\text{Ind}(S)&=\{i/\exists z(\phi_i(z)=1)\\ &\quad\wedge\ \forall x((\phi_i(x)=1\ \wedge\ \forall y(y<x\Rightarrow\phi_i(y)\neq 1))\Rightarrow\phi_i\subseteq\phi_x)\}\\ &=\{i/\exists z(\phi_i(z)=1)\\ &\quad\wedge\ \forall x(\phi_i(x)\uparrow\ \vee\ (\phi_i(x)\downarrow\ \wedge\ \phi_i(x)\neq 1)\\ &\quad\vee\ \exists y(y<x\ \wedge\ \phi_i(y)\downarrow\ \wedge\ \phi_i(y)=1)\\ &\quad\vee\ \forall w(\phi_i(w)\uparrow\ \vee\ (\phi_i(w)\downarrow\ \wedge\ \phi_i(w)=\phi_x(w)))))\}\in\Pi_2\end{aligned}$$

by the help of the Tarski–Kuratowski algorithm. Thus $\text{Ind}(SUL)$ cannot be $\Sigma_{2,2}$-complete since $\overline{\text{Ind}(SUL)}\leqslant_m\text{Ind}(SUL)$ means $\phi_x\in\overline{SUL}\Leftrightarrow\phi_{f(x)}\in SUL$ for some recursive function f, which gives an immediate contradiction using the recursion theorem. In conclusion $\text{Ind}(SUL)$ is an example of an index set not in $\Sigma_2\cup\Pi_2$ which is a difference of Σ_2 sets but not complete for such differences.

A Lower Bound

In this section we show that Π_1 is a lower bound for the position of index sets of identifiable sets in the arithmetical hierarchy. No identifiable set can include a nonempty function set with index set in Σ_1. Let $E = \bigcup_{n=0}^{\infty} (\mathbf{N}^2 \cup \{*\})^n$, and $s_1 = \langle a_0,..., a_n \rangle$ and $s_2 = \langle b_0,..., b_m \rangle$ be elements from E.

Then $s_1 \cdot s_2 = \langle a_0,..., a_n, b_0,..., b_m \rangle$. We call s_2 **an extension of $\mathbf{s_1}$** and write $s_1 \leqslant s_2$ if and only if there exists $s_3 \in E$ with $s_1 \cdot s_3 = s_2$. We write $s_1 < s_2$ if $s_1 \leqslant s_2$ and $s_1 \neq s_2$. For every partial recursive function f we say that **$\mathbf{s_1}$ is contained in f** (notation: $s_1 \subseteq f$) if and only if $\forall 0 \leqslant i \leqslant n(a_i = (x, y) \Rightarrow f(x) = y)$.

If $s_1 \subseteq f$ and $\mathbf{f} = \langle e_0, e_1,... \rangle$ is an enumeration of f we define the enumeration $s_1 \cdot \mathbf{f}$ of f by $s_1 \cdot \mathbf{f} = \langle a_0,..., a_n, e_0, e_1,... \rangle$.

Let $C = \{s/s \in E$ and s is contained in some p.r. function$\}$. Then for every INM m and $c = \langle a_0,..., a_n \rangle \in C$ define $m[c]$ to be the last output of m after inspecting $a_0,..., a_n$ in this order and before requesting the next input.

Let $(c_i)_{i=0}^{\infty}$ be an effective enumeration of C. Define for every p.r. function h and for every $n \in N$ such that $\forall x \leqslant n(h(x)\downarrow)$, the finite sequence c_h^n by $c_h^0 = c_{h(0)}$ and $c_h^{i+1} = c_h^i \cdot c_{h(i+1)}$. In the case where h is recursive we get an infinite sequence c_h^{∞} continuing this process infinitely often.

LEMMA 1. *For every identifiable set M of p.r. functions and for every $x \in N$, $\{g \in \mathbf{P}/c_x \subseteq g\} \nsubseteq M$.*

Proof. Consider $M \subseteq \mathbf{P}$ identifiable and an INM m with $M_m = M$. We proceed by contradiction. Assume that there exists $x \in \mathbf{N}$ such that

$$X = \{g \in \mathbf{P}/c_x \subseteq g\} \subseteq M.$$

Then $m[\mathbf{g}]\downarrow$ for every enumeration $\mathbf{g}$ of $g \in X$. Define the function h by $h(0) = x$ and $h(n+1) = \mu j(c_h^n \cdot c_j \in C$ and $m[c_h^n] \neq m[c_h^n \cdot c_j])$. To see that h is recursive assume the contrary. Consider the minimal n with $h(n+1)\uparrow$. Then, for every z, $c_h^n < c_z$ implies $m[c_h^n] = m[c_z]$. Consider $Y = \{g \in \mathbf{P}/c_h^n \subseteq g\}$. Then there exist recursive functions $f_1, f_2 \in Y$ with $f_1 \neq f_2$. Let $\mathbf{f}_1$ and $\mathbf{f}_2$ be enumerations of f_1 and f_2. Then $c_h^n \cdot \mathbf{f}_1$ is an enumeration of f_1 and $c_h^n \cdot \mathbf{f}_2$ is an enumeration of f_2, and $m[c_h^n \cdot \mathbf{f}_1]\downarrow i$ and $m[c_h^n \cdot \mathbf{f}_2]\downarrow i$ where $i = m[c_h^n]$.

So we can conclude since $f_1 \neq f_2$ that not both f_1 and f_2 are elements from M_m. But $c_x = c_h^0 < c_h^n$ implies $c_x \subseteq f_1$ and $c_x \subseteq f_2$. Thus $f_1, f_2 \in X$ which is a subset of M_m by our assumption—a contradiction. Since h is recursive c_h^{∞} is defined and by construction it is an enumeration of some $f \in \mathbf{P}$. $c_x = c_h^0 < c_h^{\infty}$ implies $f \in M_m$. But $m[c_h^{\infty}]\uparrow$ by construction of h—a contradiction of $f \in M_m$. ∎

By Lemma 1 we get the following theorem.

THEOREM 3. *For every identifiable set M of p.r. functions, $M' \neq \phi$ and $\mathrm{Ind}(M') \in \Sigma_1$ implies $M' \nsubseteq M$.*

Proof. By the Rice–Shapiro Theorem (Rogers, 1967) $\mathrm{Ind}(M') \in \Sigma_1$ if and only if $\exists f$ recursive: $M' = \{g \in \mathbf{P} / \exists x \in f(\mathbf{N})(c_x \subseteq g)\}$. Thus, if $M' \neq \phi$ and $\mathrm{Ind}(M') \in \Sigma_1$, $\{g \in \mathbf{P} / c_x \subseteq g\} \subseteq M'$ for some $x \in \mathbf{N}$. By Lemma 1 we conclude $M' \nsubseteq M$. ▮

By the last result index sets of identifiable sets of functions are not in Σ_1. In Π_1 it is already possible to locate such sets. Consider, for example, the set $F = \{f \in \mathbf{P} / \forall x (f(x)\downarrow \Rightarrow f(x) = 0)\}$. Then $\mathrm{Ind}(F) \in \Pi_1$, and of course F is (sharply) identifiable.

Inductive Inference Machines with Oracles

In the same way as oracle machines are defined as modified forms of Turing machines, inductive inference machines with oracles may be introduced. For a given set A an INM with oracle A (AINM) works like an INM defined in the usual way; but in addition to the operations performed by an INM, an AINM may require obtaining an answer to questions of the form "$x \in A$?." All notations and definitions about inductive inference machines given in the first part of this paper transfer to inductive inference machines with oracles.

Let $\mathbf{HALT} = \{(i, x) / \phi_i(x)\downarrow\}$ denote the halting problem. It is not difficult to see that there exists a **HALT**-INM m with $\mathbf{P} \subseteq M_m$: in every stage n, m requests an input a_n and outputs the least i such that $\langle a_0, ..., a_n \rangle$ is contained in ϕ_i. More interesting is the question if an oracle A with $\mathbf{HALT} \leqslant_{\mathrm{T}} A$ is necessary to get $P \subseteq M_m$ for some AINMm. We shall answer this question affirmatively.

First, we prove some basic facts. In the following, d is the partial recursive function defined by $d(x) = \phi_x(x) + 1$.

LEMMA 2. *For every p.r. function d' with $d \subseteq d'$,* $\mathrm{Ind}(d') \cap \mathrm{Dom}(d') = \phi$.

Proof. Suppose that there exists some $x \in \mathrm{Ind}(d') \cap \mathrm{Dom}(d')$. Then $\phi_x(x) = d'(x)$ and $\phi_x(x)\downarrow$. Therefore $d(x) = \phi_x(x) + 1 = d'(x) + 1$, a contradiction to $d \subseteq d'$. Thus, $\mathrm{Ind}(d') \cap \mathrm{Dom}(d') = \phi$. ▮

LEMMA 3. *For every p.r. function d' with $d \subseteq d'$ there exists a recursive function α such that*

$$\alpha(\mathrm{Dom}(d)) \subseteq \mathrm{graph}(d)$$

and

$$\alpha(\overline{\mathrm{Dom}(d)}) \subseteq \overline{\mathrm{graph}(d')}$$

Proof. Obviously there exists a recursive function $q: \mathbf{N}^2 \to \mathbf{N}^2$ such that for every i

$$\text{range}(\lambda x . q(i, x)) = \begin{cases} \text{graph}(d') \cup (\mathbf{N} \times \{0\}) & \text{if} \quad i \in \text{Dom}(d) \\ \text{graph}(d') & \text{otherwise} \end{cases}$$

Define by the s_n^m theorem a recursive function $\beta(i)$ as the index of the p.r. function defined by

$$\phi_{\beta(i)}(x) = p_2(\mu j \colon p_1(q(i, j)) = x),$$

where p_1, p_2: $\mathbf{N}^2 \to \mathbf{N}$ are the projections to the first, respectively, to the second component.

We show that $(\beta(i), 1) \notin \text{graph}(d')$ if $i \notin \text{Dom}(d)$ and $(\beta(i), 1) \in \text{graph}(d)$ if $i \in \text{Dom}(d)$. Then $\alpha(i) = (\beta(i), 1)$ is the desired function.

Consider $i \notin \text{Dom}(d)$. Then $\text{range}(\lambda x . q(i, x)) = \text{graph}(d')$, i.e., $\phi_{\beta(i)} = d'$ and therefore $\beta(i) \notin \text{Dom}(d')$ by Lemma 2. In conclusion $(\beta(i), 1) \notin \text{graph}(d')$.

Consider $i \in \text{Dom}(d)$. Then $\text{range}(\lambda x . q(i, x)) = \text{graph}(d') \cup (\mathbf{N} \times \{0\})$.

Now, we proceed by contradiction to show $\phi_{\beta(i)}(\beta(i)) = 0$. Assume the contrary. Then, $\phi_{\beta(i)}(\beta(i)) = d'(\beta(i))$. On the other hand, $\phi_{\beta(i)}(\beta(i))\downarrow$, i.e., $d(\beta(i)) = \phi_{\beta(i)}(\beta(i)) + 1 = d'(\beta(i))$ and therefore $\phi_{\beta(i)}(\beta(i)) \neq d'(\beta(i))$—a contradiction.

Thus, $\phi_{\beta(i)}(\beta(i)) = 0$ and therefore $d(\beta(i)) = 1$, yielding $(\beta(i), 1) \in \text{graph}(d)$. ▮

Blum and Blum (1975) show that for every INM m and for every $f \in \mathbf{P}$ with $f \in M_m$ there exists some sequence $c \in C$ with $c \subseteq f$ and $m[c] = m[c']$ for every extension c' of c contained in f. The proof transfers directly to INM's with oracles. (Assuming the contrary we can exhibit an A-effective enumeration of some $f \in M_m$ with $[\mathbf{f}]\uparrow$, where A is the oracle of m.)

In the following we call $c \in C$ m-**convergent** for $f \in \mathbf{P}$ iff $c \subseteq f$ and $\forall c'(c < c' \wedge c' \subseteq f \Rightarrow m[c] = m[c'])$.

By the above considerations, for every A-INM m and for every $f \in M_m$ there exists some $c \in C$ which is m-convergent for f. An INM m is called **consistent** if and only if for every $c \in C$, $m[c] = i$ yields $\phi_i(x) = y$ for all pairs (x, y) in c.

Blum and Blum (1975) show that we can construct to every INM m a consistent INM m' such that $M_m \subseteq M_{m'}$, if $\{f \in \mathbf{P}/\text{Dom}(f) \text{ is finite}\} \subseteq M_m$. Again, this proof transfers directly to INM's with oracles. Using this result we show

THEOREM 4. *If m is an A-INM with $\mathbf{P} \subseteq M_m$ then* $\mathbf{HALT} \leqslant_T A$.

Proof. By our remark we may assume without loss of generality that m is consistent. Consider $c \in C$ which is m-convergent for d. Define an A-recursive function χ by

$$\chi(x, y) = \begin{cases} 1 & \text{if} \quad m[c] = m[c \cdot \langle x, y) \rangle], \\ 0 & \text{otherwise} \end{cases}$$

Since m is consistent, $m[c \cdot \langle x, y) \rangle] = m[c \cdot \langle x, z) \rangle]$ implies $y = z$. Thus χ is the characteristic function of the graph of the partial A recursive function f defined by

$$f(x) = \mu y : \chi(x, y) = 1.$$

Let $d' = \phi_{m[c]}$. Then $d \subseteq f \subseteq d'$ because

$$\begin{aligned}(x, y) \in \text{graph}(d) &\Rightarrow m[c] = m[c \cdot \langle (x, y) \rangle] \\ &\Rightarrow (x, y) \in \text{graph}(f) \\ &\Rightarrow \phi_{m[c]}(x) = y \\ &\Rightarrow (x, y) \in \text{graph}(d').\end{aligned}$$

By Lemma 3 there exists a recursive function α with

$$\alpha(\text{Dom}(d)) \subseteq \text{graph}(d) \subseteq \text{graph}(f)$$

and

$$\alpha(\overline{\text{Dom}(d)}) \subseteq \overline{\text{graph}(d')} \subseteq \overline{\text{graph}(f)}.$$

Therefore, $\text{Dom}(d) \leqslant_m \text{graph}(f)$, where $\text{graph}(f)$ is A-recursive. In conclusion $\text{Dom}(d)$ is A-recursive, i.e., $\mathbf{HALT} \equiv_T \text{Dom}(d) \leqslant_T A$. ∎

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